Role of Local and Global Geometry in Quantum Entanglement Percolation

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Establishing long-range entanglement We want to create an entangled state between qubits on two distant nodes on a lattice. Initially, entanglement is distributed only locally. What is the best protocol? We consider only pure states and perfect quantum operations. So we are looking at the low-noise limit.

Clever protocols involve performing quantum operations to change the geometry or topology of the lattice. In previous examples, with respect to the initial lattice, the new lattice has non-decreasing coordination number and percolation threshold. In this work, we find a lattice transformation that (1) allows a better protocol than the untransformed lattice. And (2) has a lower coordination number and higher classical percolation threshold.

1. One-dimensional lattice

A simple lattice: A one dimensional chain. There are two qubits on each node. Each node is entangled with its nearest neighbors. These initial states are identical **partially entangled pure states** $|\alpha\rangle$. We call them **links**. We want to entangle qubits at each end of the chain. An easy protocol is 1) probabilistically convert each entangled state to a **singlet** (i.e. maximally entangled state). This is called a **singlet conversion**. 2) Perform **entanglement swapping** at each node.

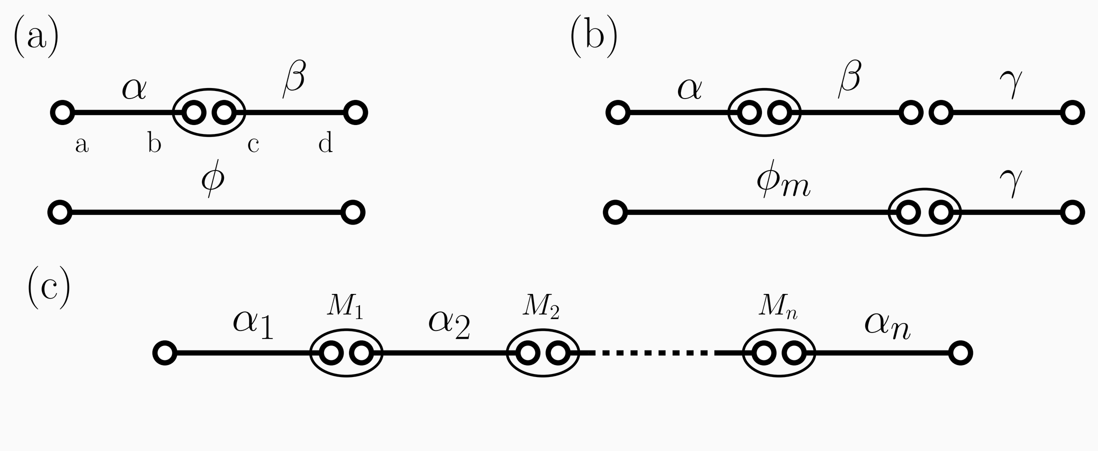


Fig. 1: Quantum repeater, a simplfied cartoon.

(a) shows a single entanglement swapping operation. Assume we have converted all links to singlets. So nodes a and b are in a singlet, and likewise with c and d. After the swap, a and d are in a singlet state. So we've distributed entanglement from nearest neighbors to next nearest neighbors. (b) Swapping at the next link in the chain. (c) After repeated swappings, the two ends of the chain are entangled.

Problem: Entanglement swapping between two singlets succeeds with probability one. But, any singlet conversion may fail and we get not a singlet, but rather a separable state, which is worthless. So one failed conversion breaks the chain. As the length of the chain increases, the probability for this to happen rapidly approaches 1. A lot of work has gone into overcoming this problem. But we don't address that here.

2. Higher-dimensional lattice

In higher dimensions, we can perform the same protocol as in Fig. 1. We just choose one path between two distant nodes and swap as if it were a 1d chain. But the situation is much better, because each node shares more than one link. Even if the singlet conversion of one link leaving a node has failed, it may have succeeded at one of the other links. Then we can follow the good link instead. Think of a tree-like lattice. If the average number of successful conversions of links leaving a node is greater than 1, the number of possible good paths grows exponentially on average with distance, so that we have a chance for long-range entanglement. The chance for a successful conversion p increases with the amount of initial entanglement per link. We use p as the **measure of entanglement** for the initial entanglement. Thus, to have a chance for long-range entanglement, p must be greater than a threshold

$$p_c = 1/(Z-1), \tag{1}$$

where Z is the coordination number. If $p(Z-1) < 1$, then the number of paths decreases exponentially on average with distance, and we have no chance of long-range entanglement.

Percolation

On a tree there is a threshold p_c, such that for $p > p_c$ there is an exponentially growing number of possible paths. This threshold also exists for lattices with loops, such as the kagome and square lattices. But, for lattices with loops, p_c is not given by (1). In fact, it may be very difficult to calculate. p_c is called the **percolation threshold**.

Singlet conversion and swapping probabilities

$|\alpha\rangle$ has entanglement p. We can convert it to a singlet with **singlet conversion probability** p. **It turns out that the probability of getting a singlet from entanglement swapping with two copies of $|\alpha\rangle$ is also p.** This is very important in the following. Suppose we operate on all links, doing singlet conversions on some links and swappings on pairs of other links. Then the result will be a lattice of links each of which is in the singlet state with probability p and separable with probability $1-p$.

3. Lattice Transformations

Good. This convert-then-swap protocol has non-zero probability of succeeding, that is, of achieving long-range entanglement. But could there be another protocol that gives a higher probability ?

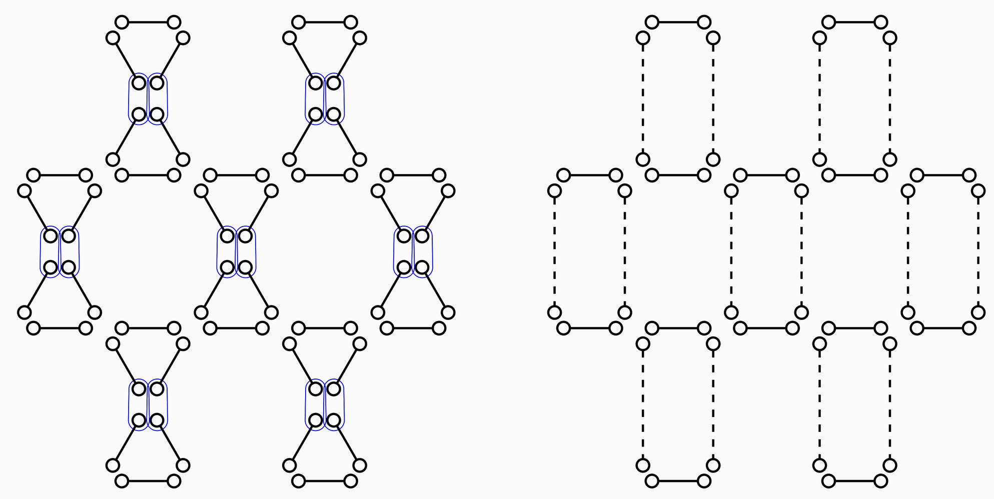


Fig. 2: Lattice conversion: Kagome to square.

The answer is yes. For example, in the left side of Fig. 2 we have a kagome lattice. Instead of immediately performing singlet conversions, we perform swappings on qubits circled in blue, that is, on the vertical pairs. Then we do a singlet conversion on the remaining horizontal links The result, shown on the right, is a square lattice. And from the discussion in the previous box, we see that each link is a singlet with probability p. Now, for the square lattice $p_c = 1/2$, and for the kagome lattice $p_c \approx 0.52$ So for an initial amount of entanglement p such that $1/2 < p < 0.52$, we can have long range entanglement via swapping on the square lattice, but not via swapping on the kagome lattice. In this sense, the transformation gives a better protocol.

4. Structure of the transformed lattice

Notice in Fig. 2 that both the kagome and square lattices have coordination number $Z = 4$. That is, Z did not decrease under the transformation. Also $p_c = 1/2$ for the square lattice, which is smaller than $p_c \approx 0.52$ for the kagome lattice. That is, the classical critical threshold decreased under the lattice transformation. Both of these properties seem reasonable, because they mean that the transformed lattice is more highly connected, or at least not less so. Since getting a higher chance for an infinitely long path is our goal, looking for a protocol giving a more highly connected lattice seems like the right strategy. But are these properties necessary for an improvement over the simple singlet conversion and swap ? Is there an advantageous transformation protocol such that Z decreases and p_c increases? In the following, we see that the answer is yes. Obviously, this will require concentrating entanglement on the links of the transformed lattice We do this by using an operation that we call **partial entanglement swapping**. The example we use to show this (at the top of the third column) transforms the triangular lattice into the hexagonal lattice.

5. More about swapping and purification

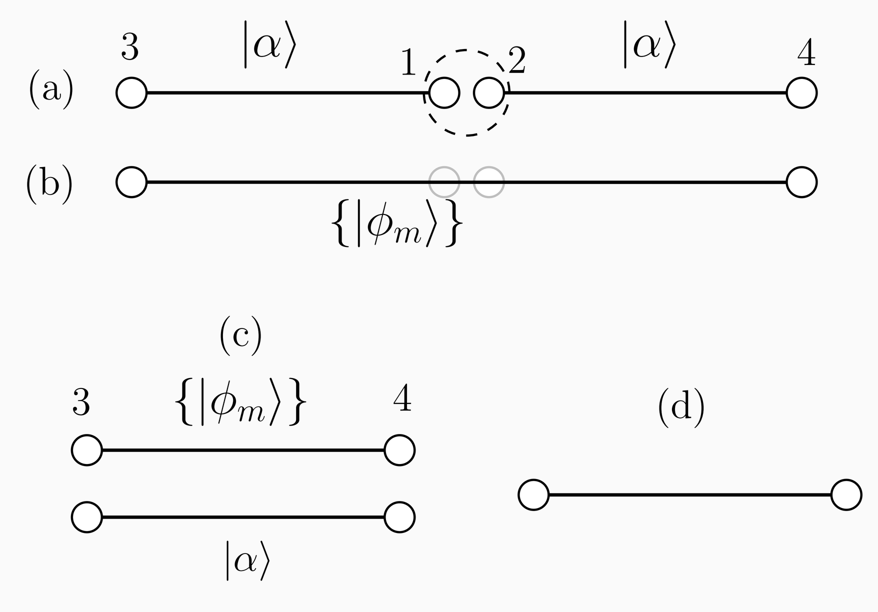


Fig. 3: Swapping and purification

Swapping is a composite operation. Here are the components: (a) First, perform a Bell measurement on 1 and 2. The result (b) is one of four states $\{|\phi_m\rangle\}$ on 3 and 4. Two outcomes are already singlets. For the two outcomes that are not singlets, we perform a singlet conversion. Averaging over the outcomes of both the Bell measurement and the conversions, we get the probability for the entire swapping operation to result in a singlet between 3 and 4.

Partial entanglement swapping

Consider **protocol** A: If the outcome $|\phi_m\rangle$ is partially entangled, then we stop after the Bell measurement in Fig 3a, and do not proceed with singlet conversion on $|\phi_m\rangle$. In Fig 3c we then use a state $|\alpha\rangle$ (not shown in (a) and (b)), that is arranged in parallel, to perform a **purification** with $|\phi_m\rangle$. This results in a state (d) that may be more highly entangled than either of $|\phi_m\rangle$ and $|\alpha\rangle$.

Why do we do this partial entanglement swapping? Consider **protocol** B: We instead do the full entanglement swapping so that we get either a singlet or a useless separable state in parallel with $|\alpha\rangle$. The goal is to get a singlet on 3 and 4. If the full swapping failed, then the only way to use $|\alpha\rangle$ is to try a singlet conversion on it. It turns out that, depending on p, the probability of success in protocol A (partial entanglement swapping) may be higher than in protocol B (full entanglement swapping).

6. Triangular to hexagonal lattice

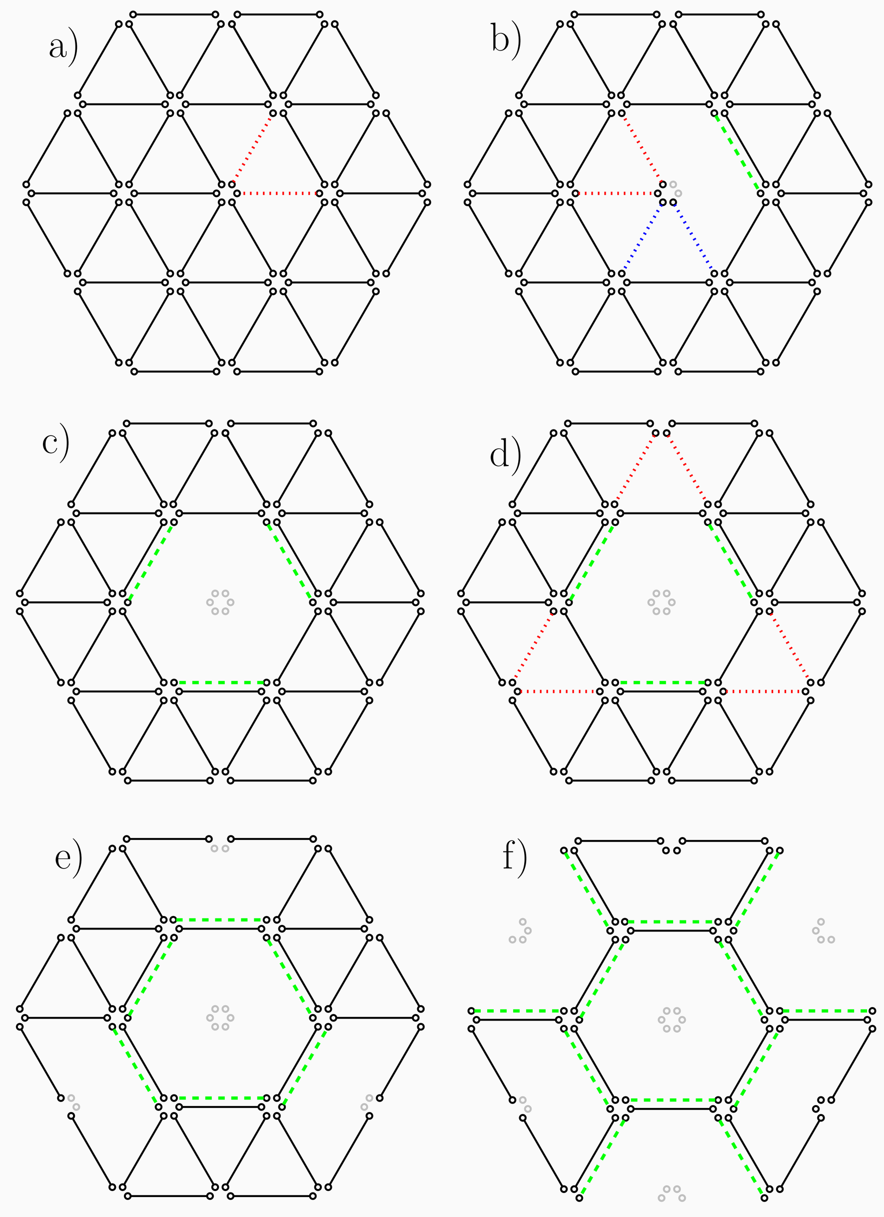
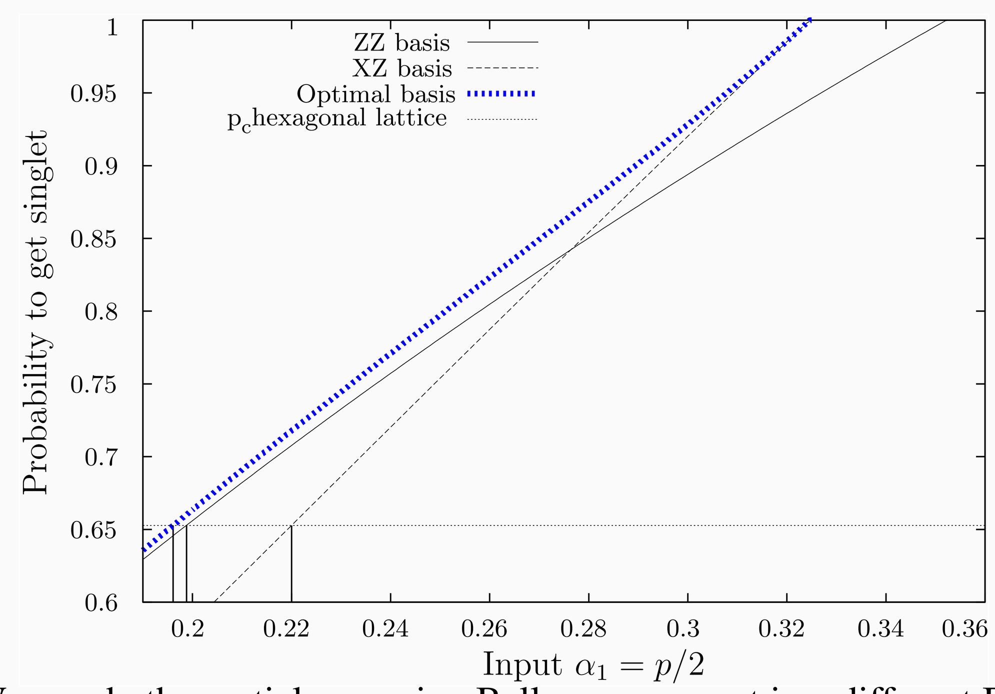


Fig. 4: Triangular to hexagonal.

Transforming triangular to hexagonal lattice via partial entanglement swapping. Initially, each line is a link in the partially entangled state $|\alpha\rangle$ (a) Bell measurement on two central qubits on dotted red links. (b) Resulting green dashed link is $\{|\phi_m\rangle\}$. Blue and red dotted links show the next Bell measurements. (c) Green dashed links are resulting partial swapping outcomes. (d), (e), (f) and so on .... In (f), all the black links are $|\alpha\rangle$ and all the green links are one of $\{|\phi_m\rangle\}$. Wherever $|\phi_m\rangle$ is a singlet, we do nothing because we have what we want. Wherever $|\phi_m\rangle$ is partially entangled we purify $|\alpha\rangle$ and $|\phi_m\rangle$ as in Fig. 3.

Now we have transformed the triangular lattice ($Z = 6$, $p_c \approx 0.35$) into the hexagonal lattice ($Z = 3$, $p_c \approx 0.65$). We computed the probability with this protocol to get a singlet from each double link on the hexagonal lattice with the following result. Let $\alpha_0 = 1 - p/2$. If α_0 is less than the real root $\alpha_0^* \approx 0.6478$ of $\alpha_0^3 - \alpha_0^2 + \alpha_0 - 1/2 = 0$ then a singlet is obtained with probability 1. This corresponds to the condition $p > p^* \approx 0.70$. Compare this to percolation on the original triangular lattice, which is deterministic only for $p = 1$.

7. Optimize probability to get singlets on hexagonal lattice



We can do the partial-swapping Bell measurement in a different Bell basis. That is, we rotate the basis for each of the two qubits used to define the Bell basis. Up to now, we have implicitly considered the unrotated ZZ basis. We also calculate the probability to get singlets on the hexagonal lattice with the XZ basis, and furthermore optimize over all rotations for each p numerically. The figure shows that, in general, this improves the probability for getting a singlet, but does not move the upper threshold for getting a singlet with probability 1.

What else?

Other questions regarding the relation of lattice structure to long range entanglement distribution: Here we have lowered the threshold for deterministic (i.e. with probability 1) long-range entanglement. One can also lower the critical threshold below which the probability is 0. But there are no known protocols to do this without increasing the classical connectivity as measured by Z and p_c. A much harder question: is there a minimum amount of initial entanglement, optimizing over all protocols, below which, long-range entanglement is impossible?

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