Entanglement Distribution on Complex Networks

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ICFO, Barcelona, Spain

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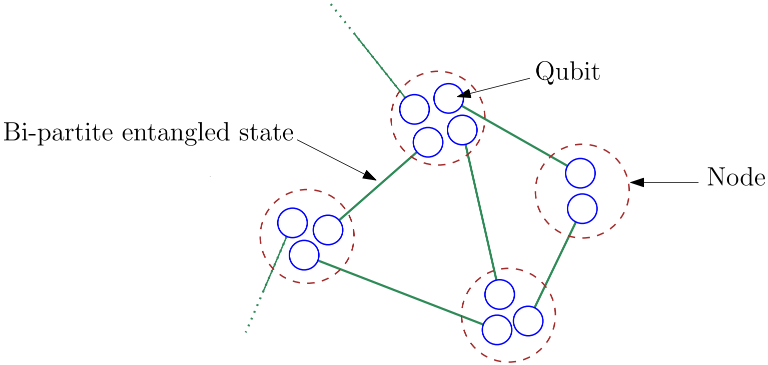
IQC workshop on quantum computation and complex networks



- *qubits* (two-level quantum systems)
- Multiple qubits and classical resources at each *node* (vertex)
- *links* (edges): bi-partite entangled (pure/mixed) two-qubit states.
- Quantum operations local: within nodes. Classical can be global. Local Operations and Classical Communication LOCC
- Goal: entangle pairs of qubits between distant nodes

- Quantum Information: Entanglement is a resource for tasks:teleportation, key distribution, fault tolerant computation
    - Creating entanglement requires local interaction. Noise increases with distance. Depolarization. Absorption. Can't distribute entanglement over long distance in a single stage!

- Long range entanglement via Network of stations or nodes that store and purify a state.
    - Generalization of quantum repeater schemes. Dür, Briegel, Cirac, Zoller, PRA 1999
    - Nodes share partially entangled states of qubits
    - Nodes(stations)/channels, Vertices/edges, Sites/bonds
    - Quantum operations probabilistic
    - Large number of random components $\Rightarrow$ Complex Networks, Percolation, Phase transition

Entanglement distribution on networks

- Given a network with a specified amount of quantum and classical resources, and a specific long range entanglement task, design the optimal protocol to achieve the task.
- E.g. Optimal: Smallest amount of resources (entanglement) per link that achieves task. Or protocol that achieves task with highest probability for a given amount of resources.
- E.g. Topology of lattice(network) may be an external constraint.
- E.g. Task: entangle fixed widely separated nodes A and B.

Entanglement: Two entangled qubits

Two entangled qubits: four-dimensional Hilbert space.

Cannot be written as a product state (in any basis).

Schmidt basis always exists for bi-partite pure state.

$$|\alpha\rangle = \sqrt{\alpha_0}\,|00\rangle + \sqrt{\alpha_1}\,|11\rangle$$

$$\alpha_0 > \alpha_1 \qquad \alpha_0 + \alpha_1 = 1 \qquad \alpha_1 \in [0, 1/2]$$

Pure, partially entangled, bipartite state

$\alpha_1 = 0$: no entanglement, $\alpha_1 = 1/2$: max. entanglement

Bell State: Singlet Conversion

Partially Entangled: $|\alpha\rangle = \sqrt{\alpha_0}\,|00\rangle + \sqrt{\alpha_1}\,|11\rangle$

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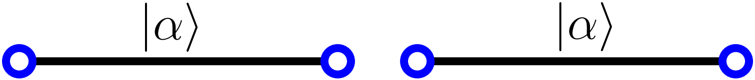
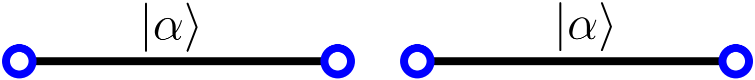
Singlet, Bell State, Maximally Entangled State Singlet Conversion Probability $p = 2\alpha_1$, for $\alpha_0 > \alpha_1$

Otherwise: product state (failure)

Distributing Entanglement

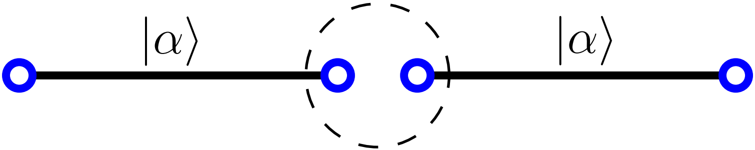


Distributing Entanglement



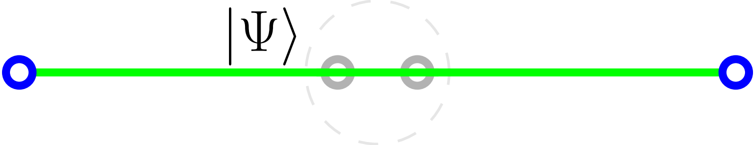
Can we entangle the two outermost qubits? Using only local operations and classical communication.

Distributing Entanglement



Yes. Entanglement Swapping.

Distributing Entanglement



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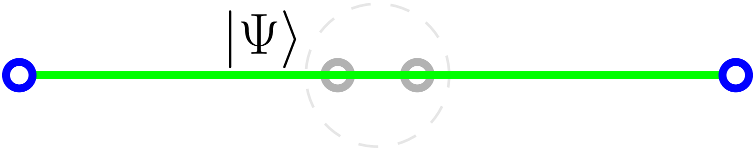
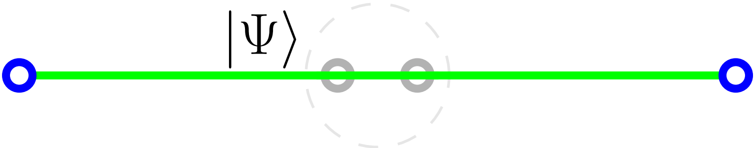


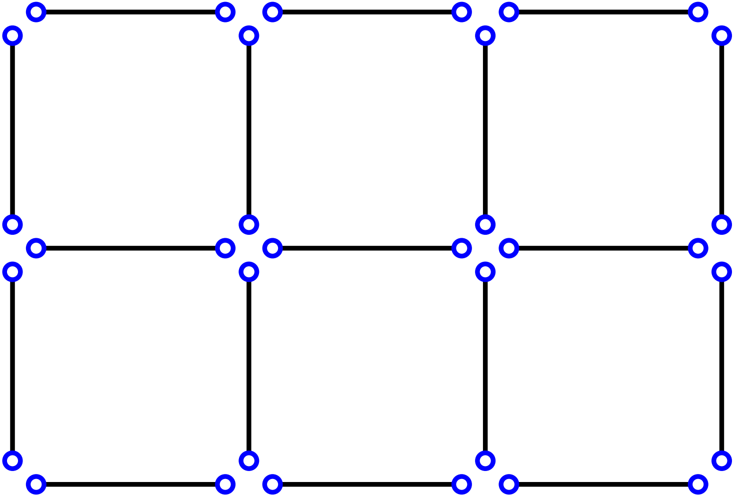
Yes. Entanglement Swapping. Get Bell state with same probability as in singlet conversion $p = 2\alpha_1$! (product state otherwise)

Distributing Entanglement



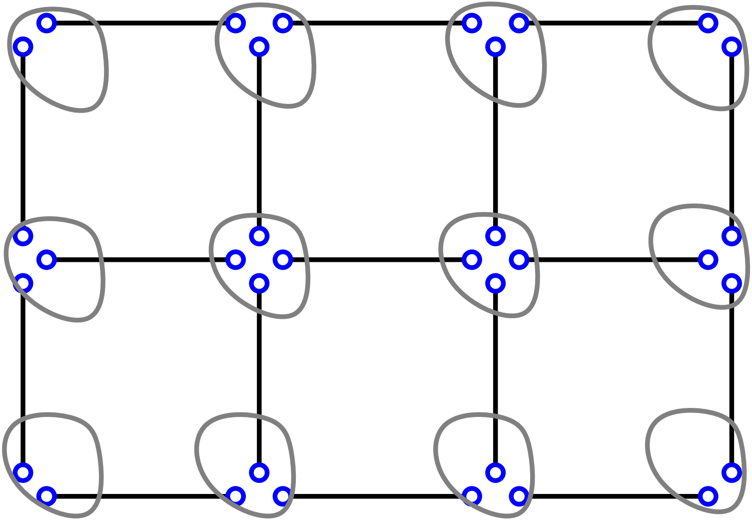
Yes. Entanglement Swapping. Get Bell state with same probability as in singlet conversion $p = 2\alpha_1$! (product state otherwise) Note: if $\alpha_1 = 1/2$, then $p = 1$.

Quantum Network



Concrete: Square lattice. Each bond is an entangled pair with amount of entanglement α_1.

Quantum Network



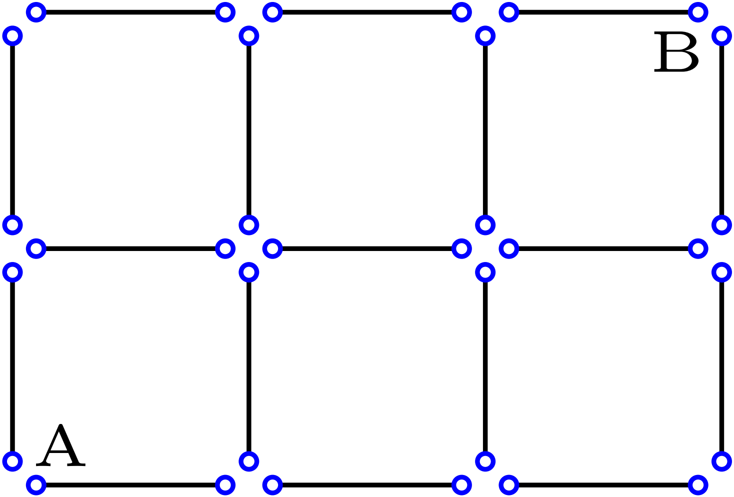
Concrete: Square lattice. Each bond is an entangled pair with amount of entanglement α_1.

Quantum Network

How to treat a network larger than two pairs. Naive method: Borrow ideas from one-dimensional quantum repeaters.

1. Attempt to put each pair in a Bell state. Here: Singlet conversion with probability of success $p = 2\alpha_1$.
2. Entanglement swappings between pairs of these Bell states. Result: New Bell state between outermost qubits, one from each of the pairs.
3. Repeat swappings, entangling ever more distant qubits.

Classical Entanglement Percolation



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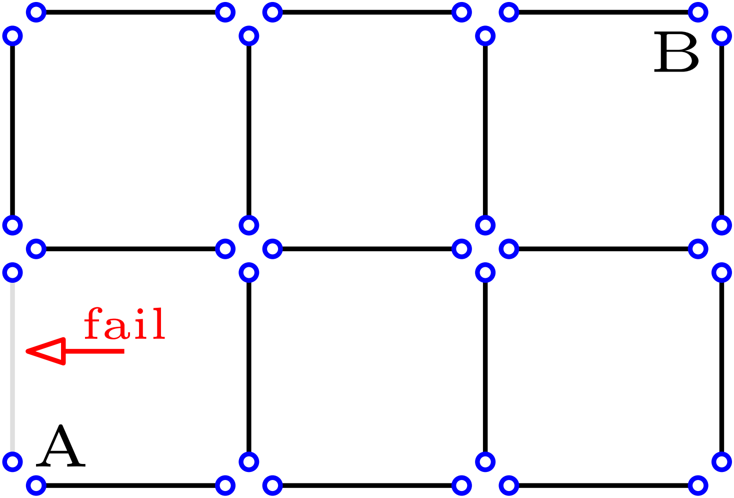

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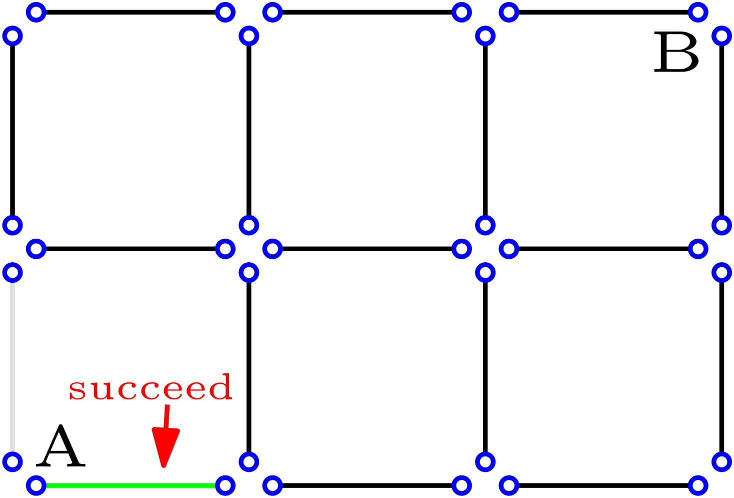

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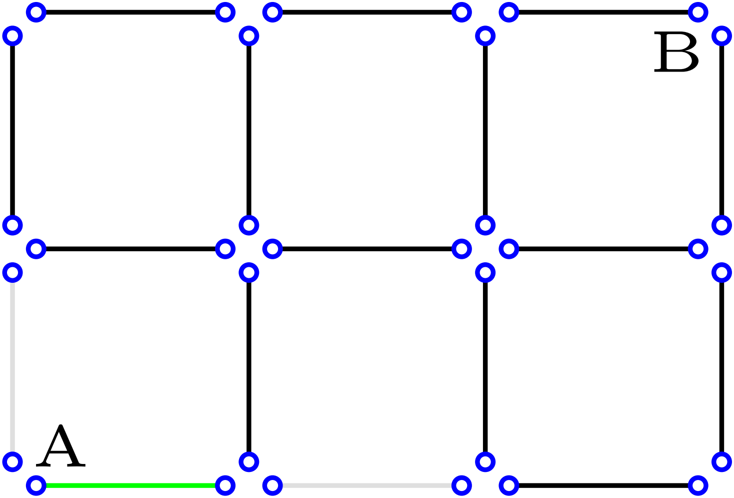

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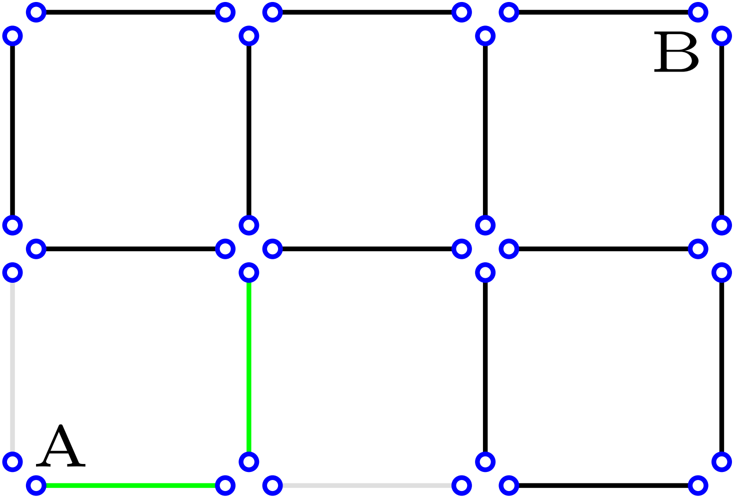

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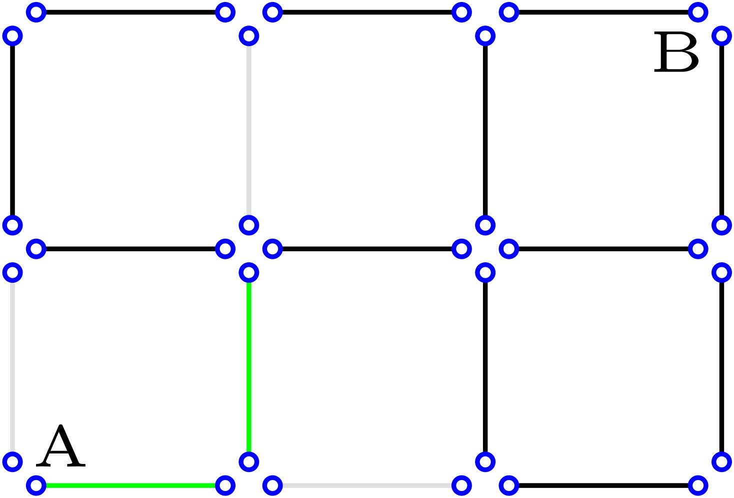

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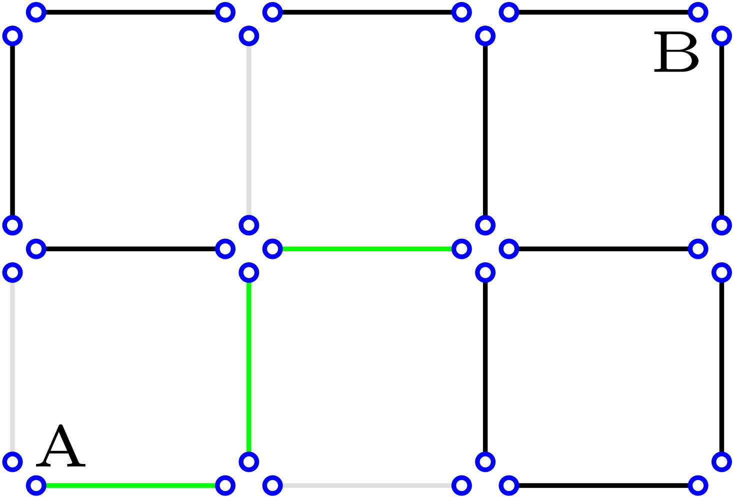

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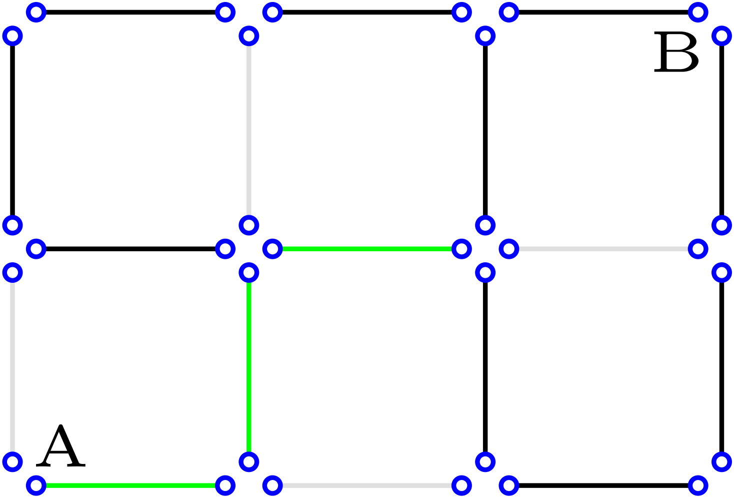

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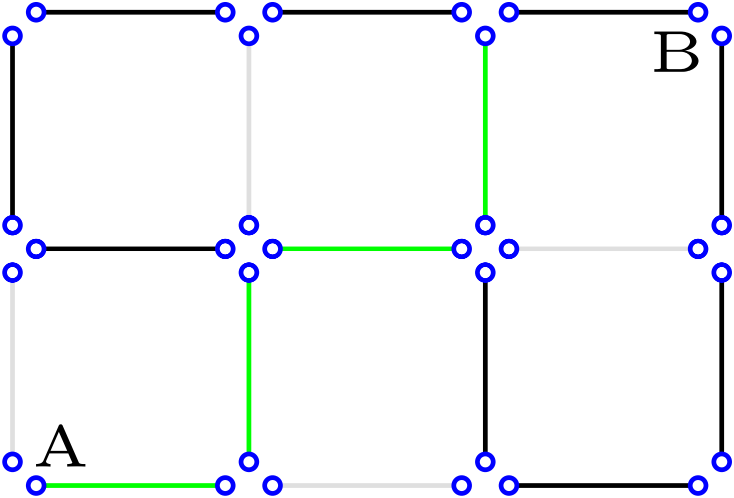

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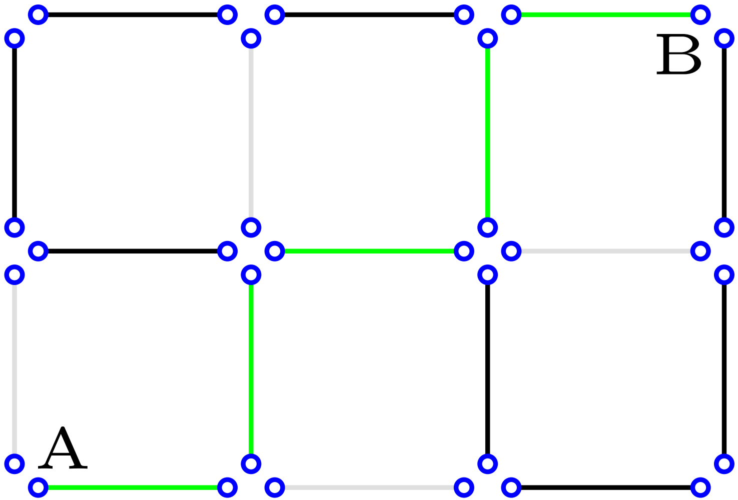

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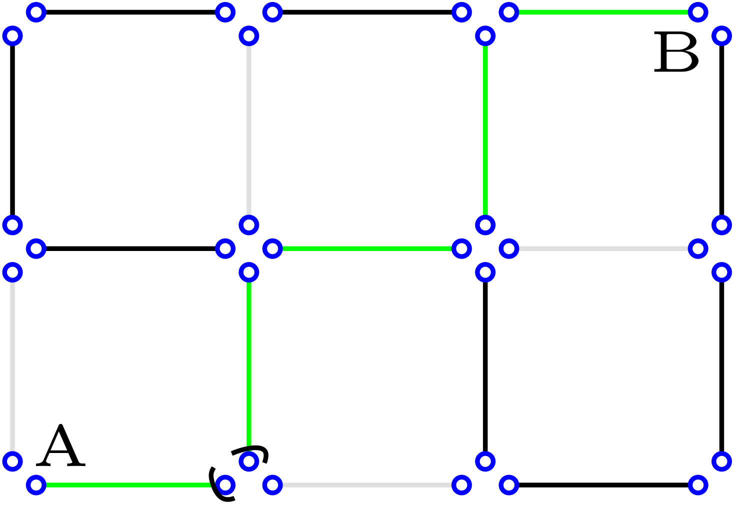

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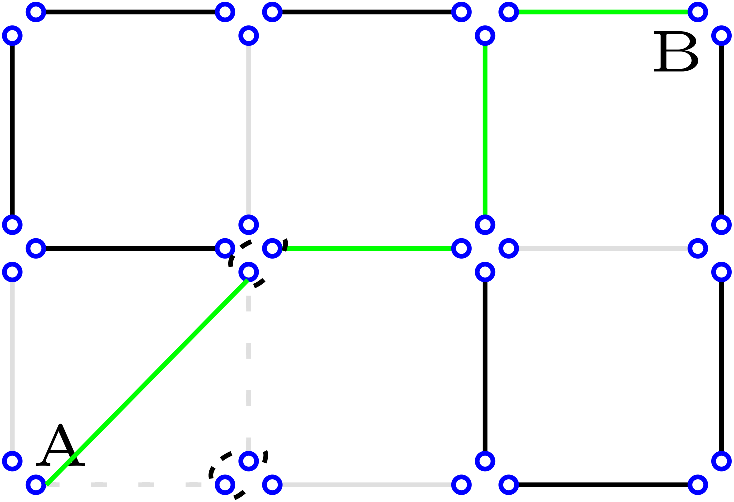

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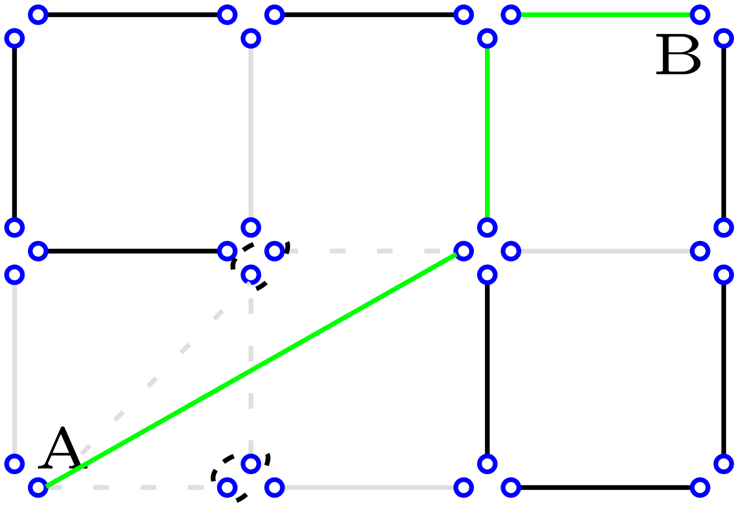

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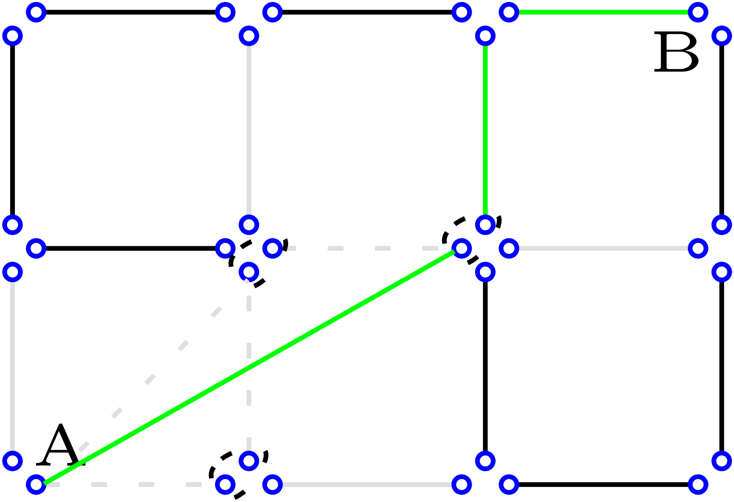

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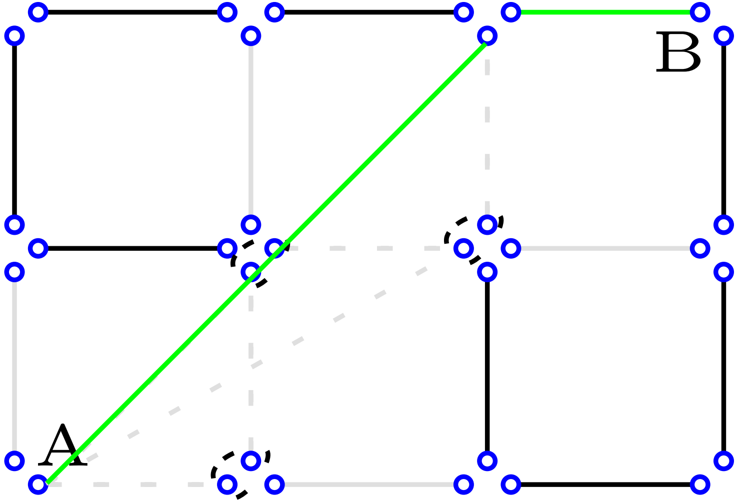

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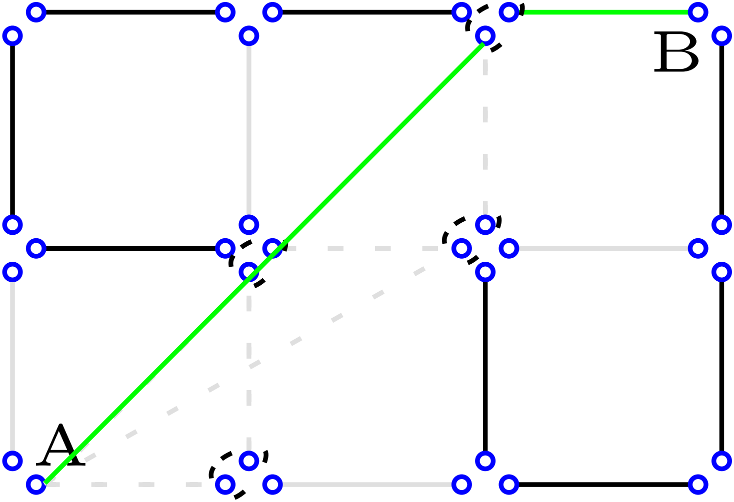

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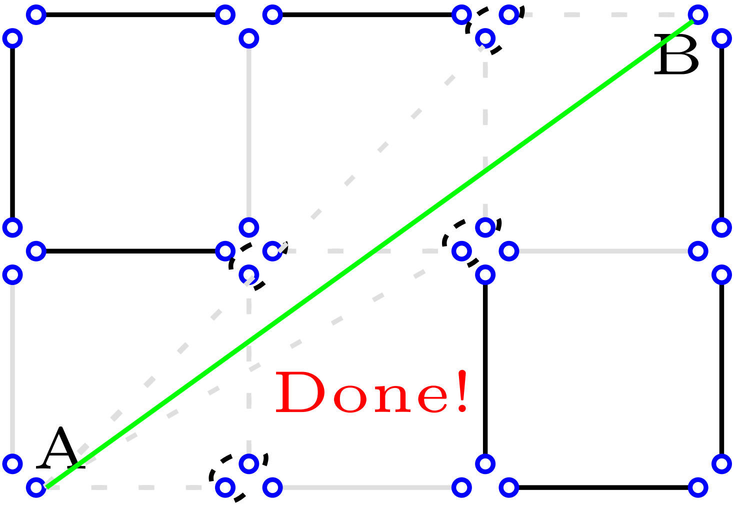
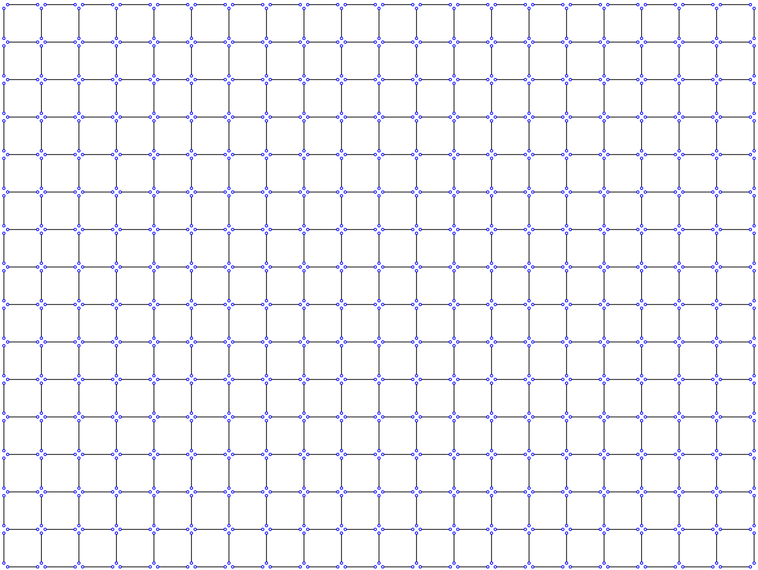


Big Network: $\alpha_1 = 0.175 \;\; p = 0.35$



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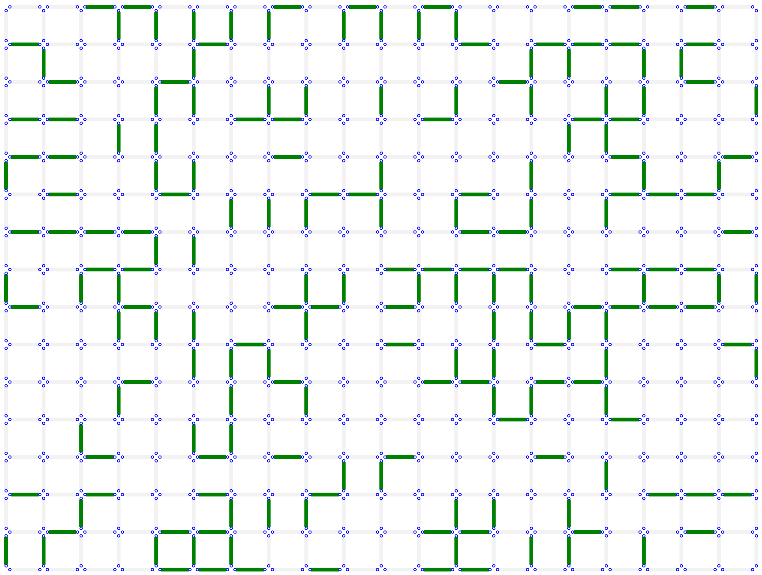

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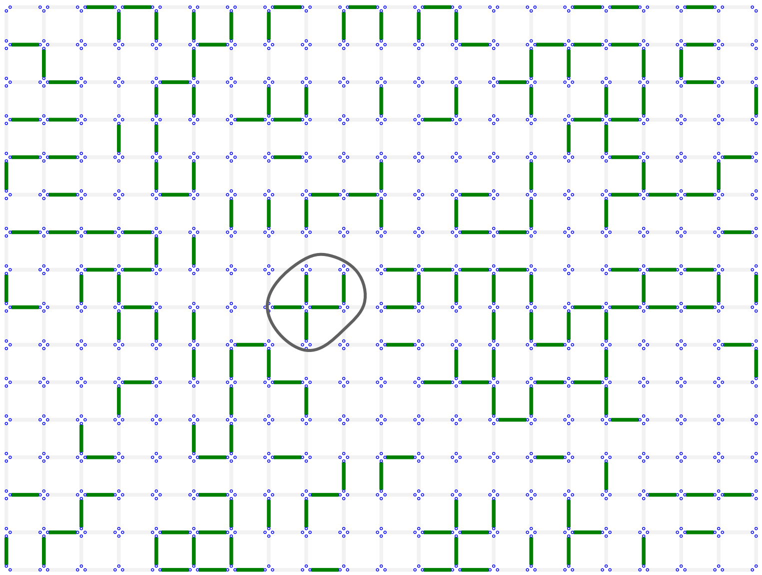

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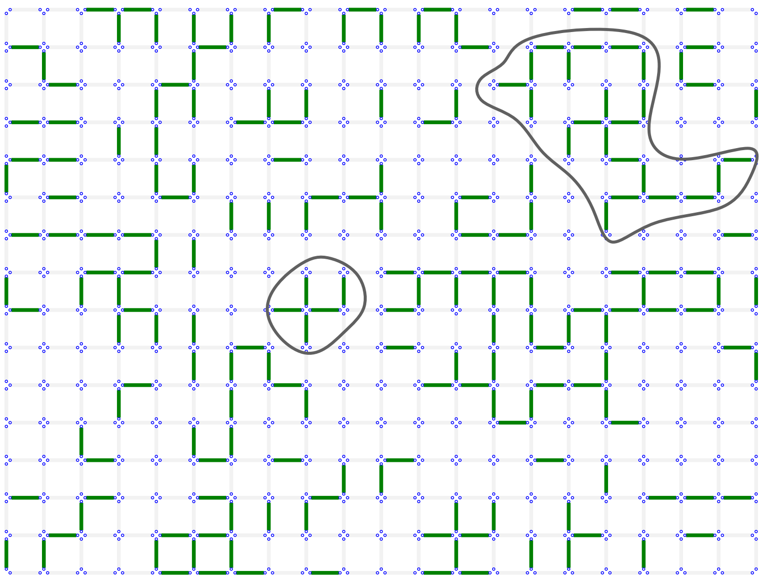

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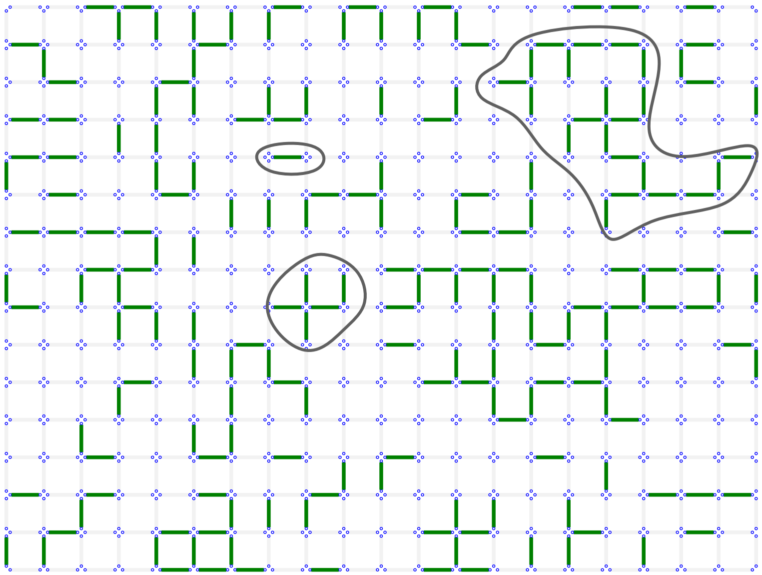

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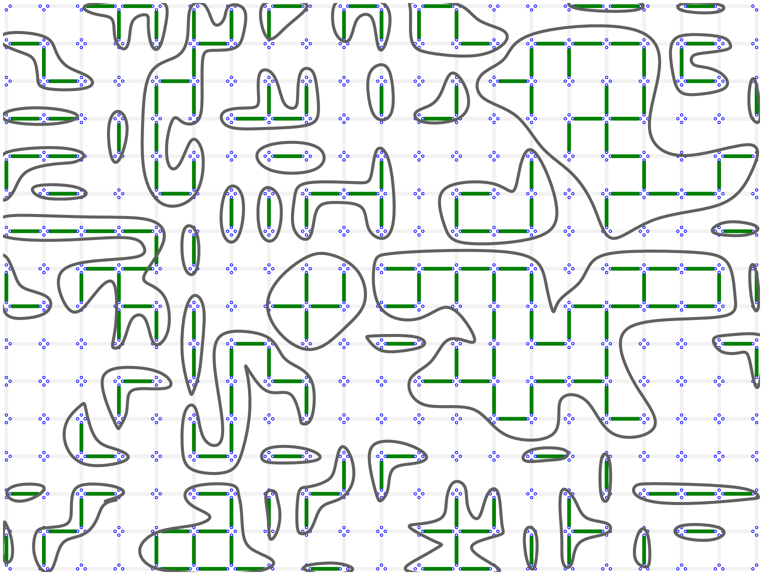

Can we do better than simply swapping along a chain ?

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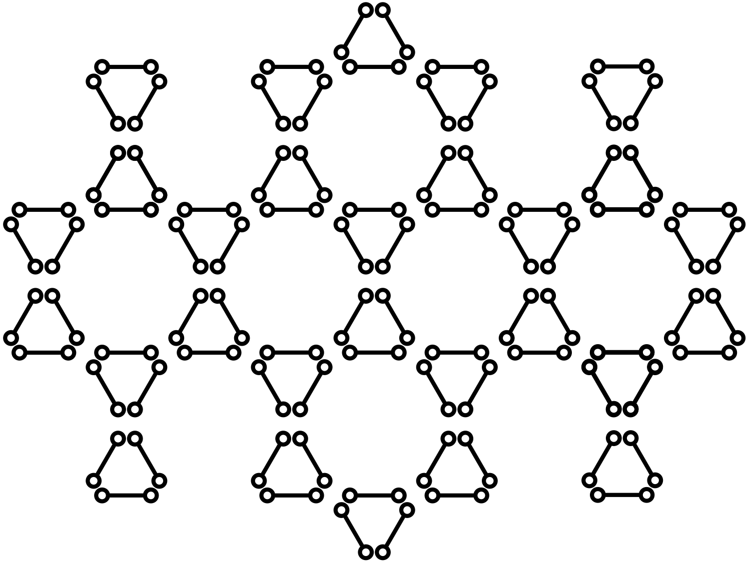
Yes. Precondition the lattice with other quantum operations. Change local structure $\Rightarrow$ Different lattice $\Rightarrow$ Different percolation threshold. Then swap along chain.

Kagome lattice



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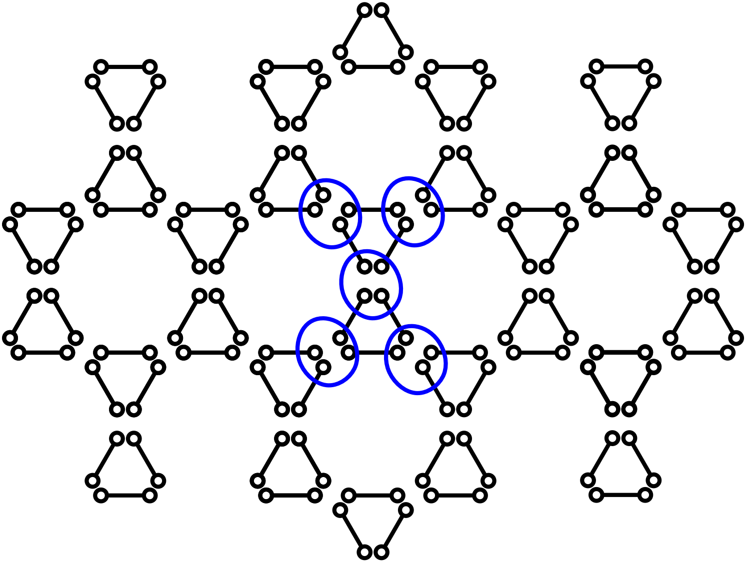

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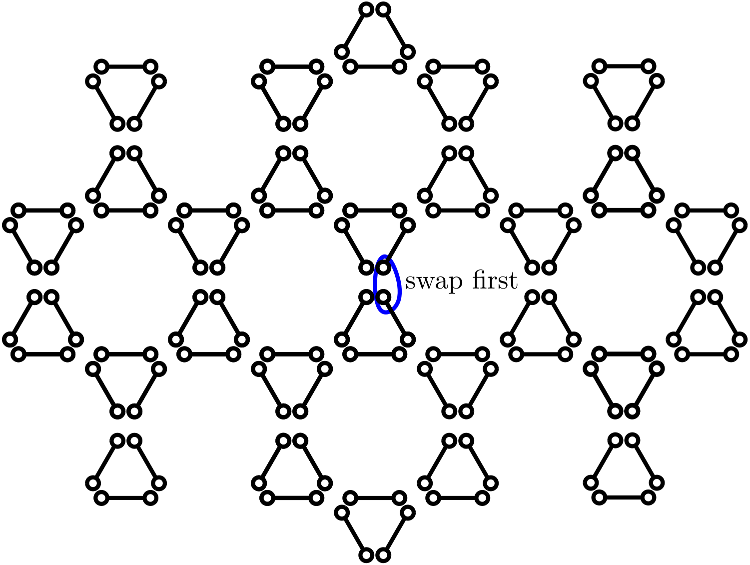

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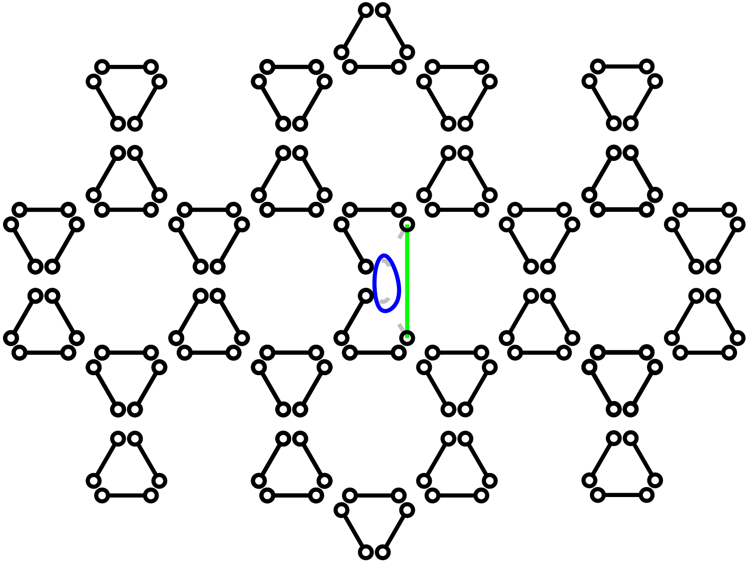

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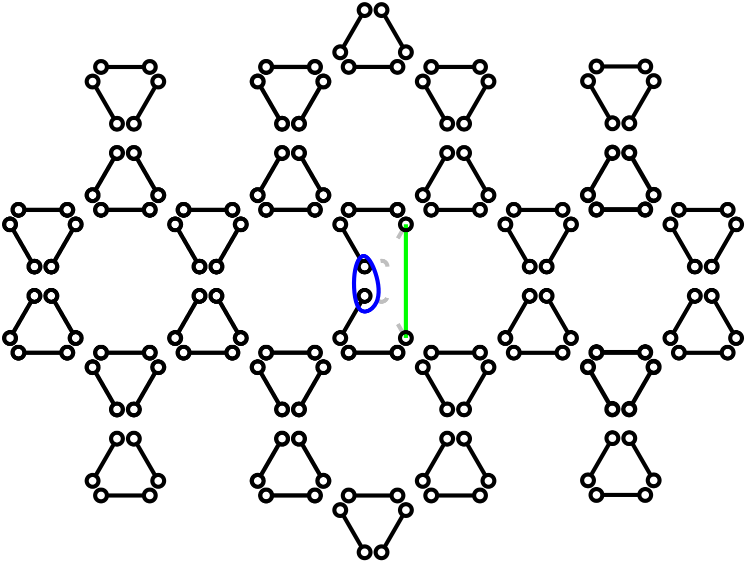

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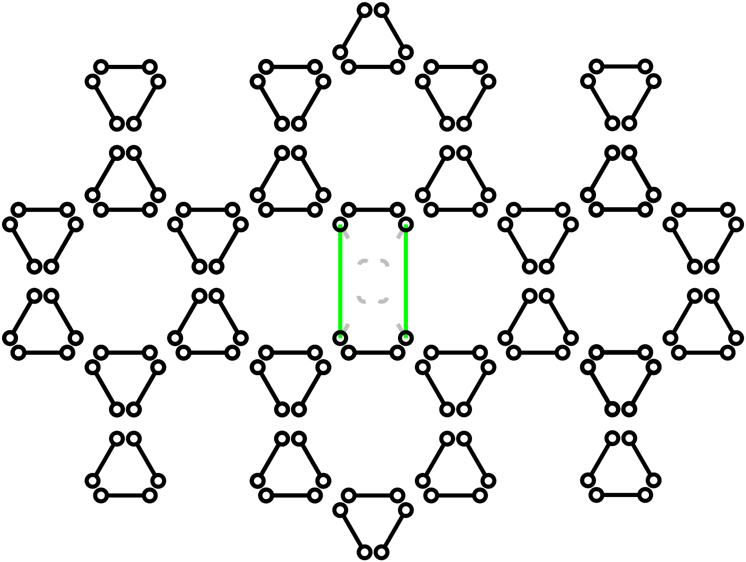
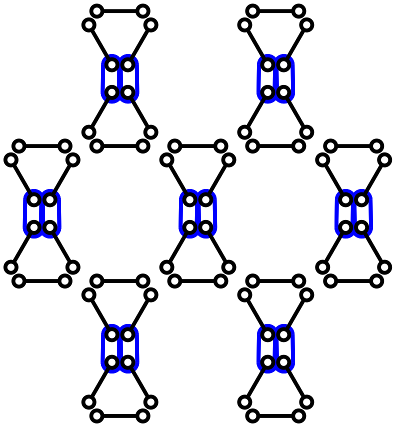
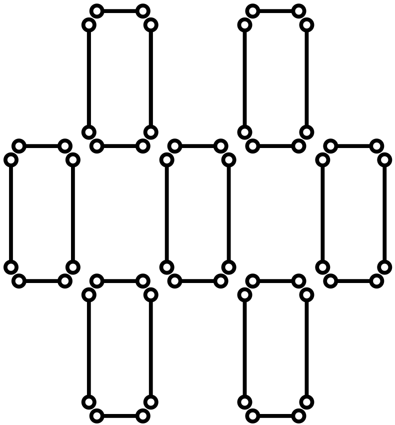


Kagome lattice to Square lattice

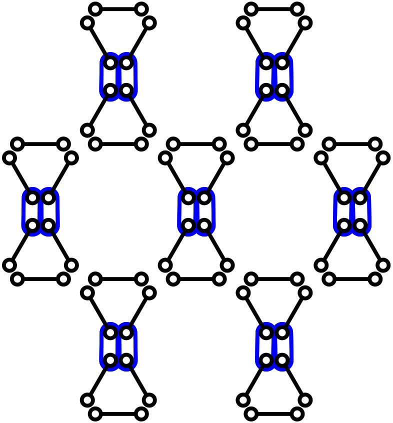




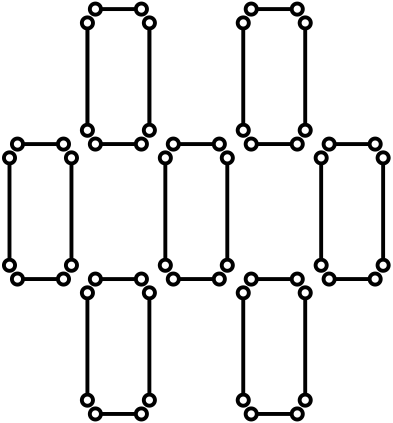
Acín, Cirac, Lewenstein, Nature Phys 2007
Perseguers, Cirac, Acín, Lewenstein, Wehr, PRA 2008

Lapeyre, Wehr, Lewenstein, PRA 2009

Kagome lattice to Square lattice



$p_c \approx 0.52$ Kagome

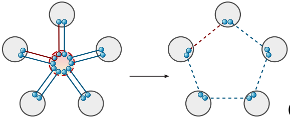


$p_c = 0.5$ Square lattice

Acín, Cirac, Lewenstein, Nature Phys 2007
Perseguers, Cirac, Acín, Lewenstein, Wehr, PRA 2008

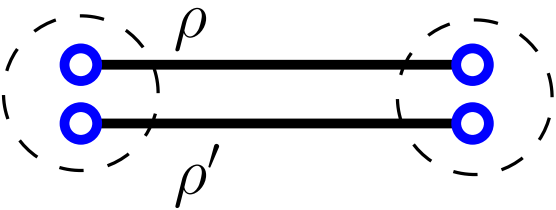
Lapeyre, Wehr, Lewenstein, PRA 2009

More entanglement percolation with pure states:

- Multipartite (GHZ) initial states ⇒ percolation on Archimedean and non-planar graphs. Perseguers, Cavalcanti, Lapeyre, Lewenstein, and Acín
- Improved swapping. Project onto larger subspace
- Conditionally complete swapping.
- Mixed states of rank ≤ 3 Broadfoot, Dorner, Jaksch, PRA 2010, EPL 2009
- Q-star transformation applied to various complex networks. E.g. for scale-free network, q-star usually advantageous when applied where degree is near mean degree.  Cuquet, Calsamiglia, PRL 2009, PRA 2011

Let's leave these and move to full-rank mixed states and complex networks.

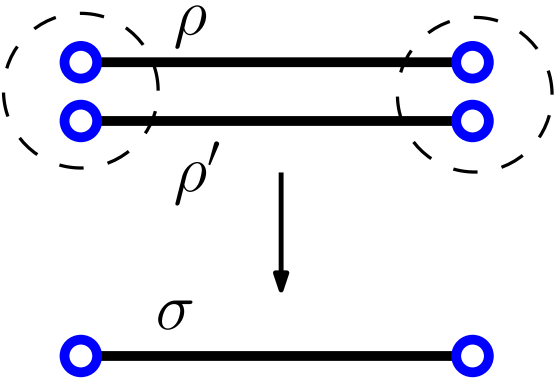
Mixed states: Entanglement Purification



Obtain σ with entanglement greater than ρ, ρ' using LOCC

(local operations and classical communication)

Mixed states: Entanglement Purification



Obtain σ with entanglement greater than ρ, ρ' using LOCC

(local operations and classical communication)

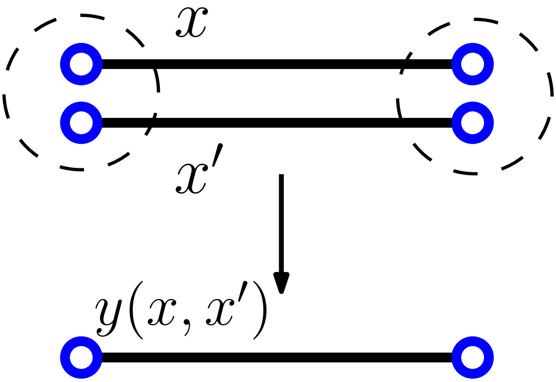
Two-qubit Mixed States

- Full-rank mixed state. All four eigenvalues positive.
- Cannot purify finite number of states to Bell pair
- Two-qubit Werner state parameterized by x

$$\rho_{\mathsf{W}}(x) = x\,|\Phi_{00}\rangle\langle\Phi_{00}| + \frac{1-x}{4}\mathbb{1}_4, \qquad 0 \le x \le 1$$

  - Werner state is a full-rank state (for $x < 1$).
  - Separable for $x \le 1/3$.
  - Concurrence: $C(x) = \max\{0, (3x-1)/2\}$. Linear, $C(\text{separable}) = 0$, $C(\text{Bell pair}) = 1$
  - Convert any state to $\rho_{\mathsf{W}}(x)$ via twirling. Can be done in lab. $\rho_{\mathsf{W}}(x)$ invariant under twirl.

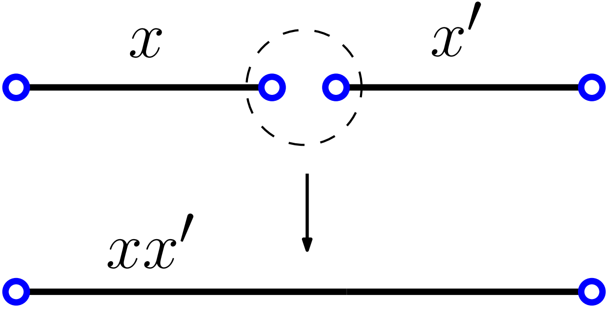
Purify Werner states. Get another Werner state.



$$y(x,x') = \frac{x + x' + 4xx'}{3 + 3xx'}, \text{ with probability } \frac{1 + xx'}{2}$$

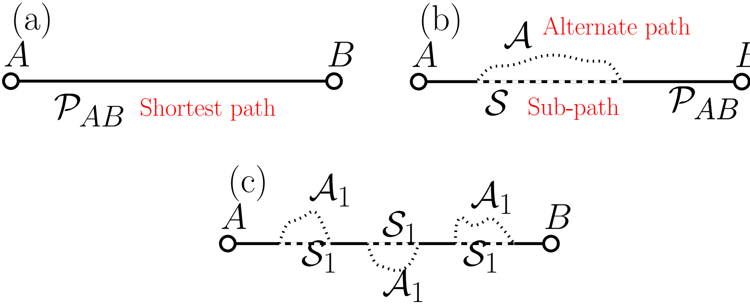
Bennett,Brassard,Popescu,Schumacher, PRL 1996 Dür, Briegel, Rep. Prog. Phys. 2007

Swap Werner states. Get another Werner state.



Entanglement increases with x; Exponential decay of entanglement with length of chain. Swapping: lose entanglement, Purification: gain entanglement.

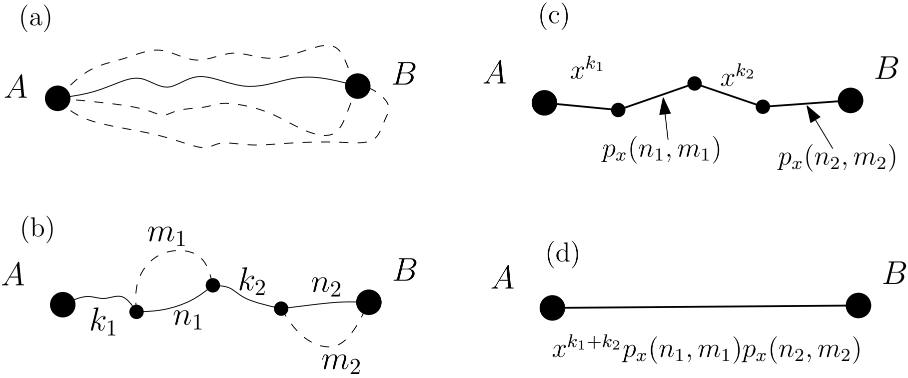
More generic network. Combine swapping and purification.

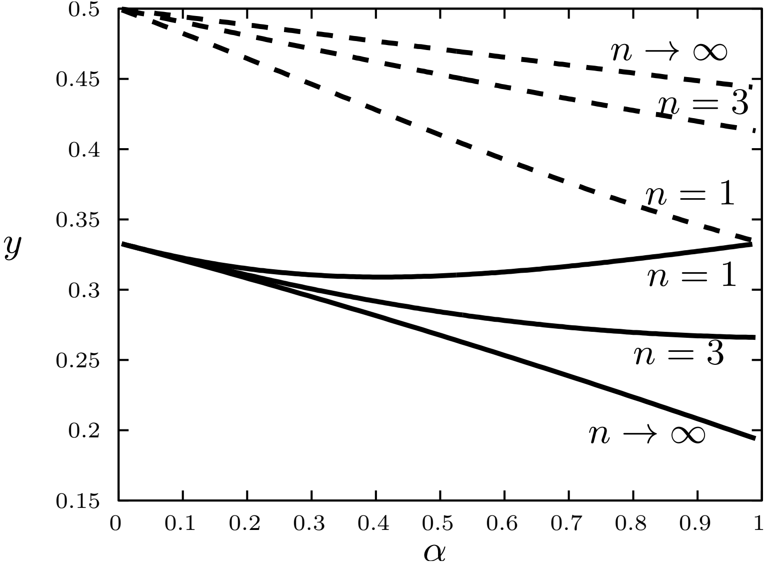


Swap first, or purify first?

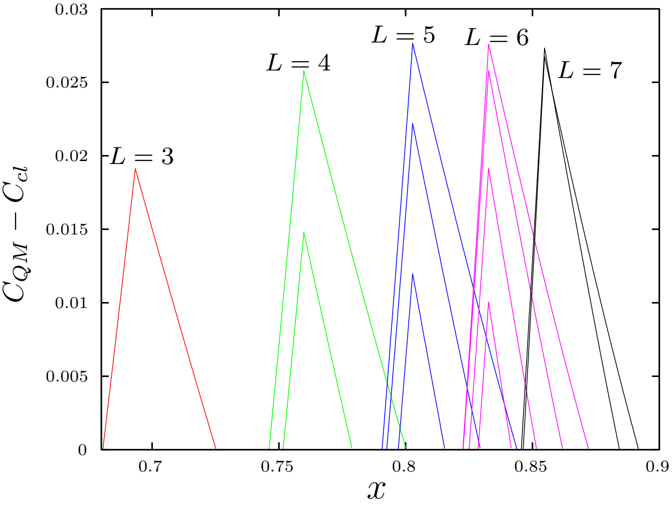
Mixed states on complex network

Werner state on each link. What is average concurrence?
(over quantum outcomes)





Multiple purifications along shortest path. $y = x^{1/L}$, α is fraction of shortest path covered by sub-paths. Lines bound range of y for which purify-swap is good.



Advantage of purify-swap depends on shortest path length L, sub-path length n, alternate path length m, Werner parameter x. Optimizing formula for gain in average concurrence is messy.

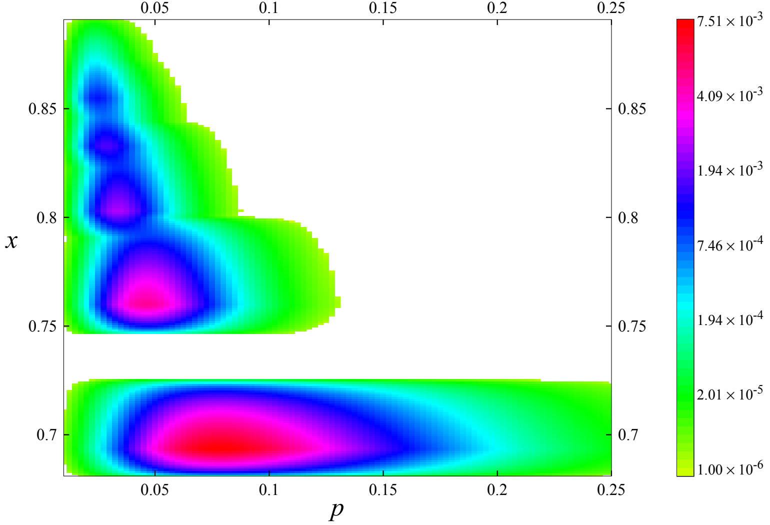
Poisson Random Graph

- Graph with N vertices. Zero or one edge between each pair. Each of the $N(N-1)/2$ edges is present with probability p.

- Density of shortest paths of length L, σ_L

$$\begin{aligned}
\sigma_1 &= p, \\
\sigma_2 &= (1-p)[1-(1-p^2)^{N-2}] \approx (1-p)\left(1-e^{-p^2 N}\right), \\
\sigma_3 &\approx \left(1-e^{-p^3(1-p)^5(N-2)(N-3)}\right)(1-p^2)^{N-2}(1-p), \text{ large } p \\
&\approx \left(1-e^{-p^3(1-p)^5 N^2}\right) e^{-p^2 N}(1-p) \\
\sigma_L &= p^L \frac{(N-2)!}{(N-L-1)!} + \mathcal{O}(p^{L+1}), \text{ small } p \\
\sigma_L &\approx \frac{1}{N} \quad \text{for } pN = 1,\ L < \text{ radius}
\end{aligned}$$

Solomonoff, Rapoport, Bull. Math. Biophys. 1948 (Who has these ?), 1951   Erdös Rényi 1959,1960,1961



Advantage of purify-swap depends on Werner parameter x and bond density of random graph p.

Monte Carlo $N = 200$, $L < 8$

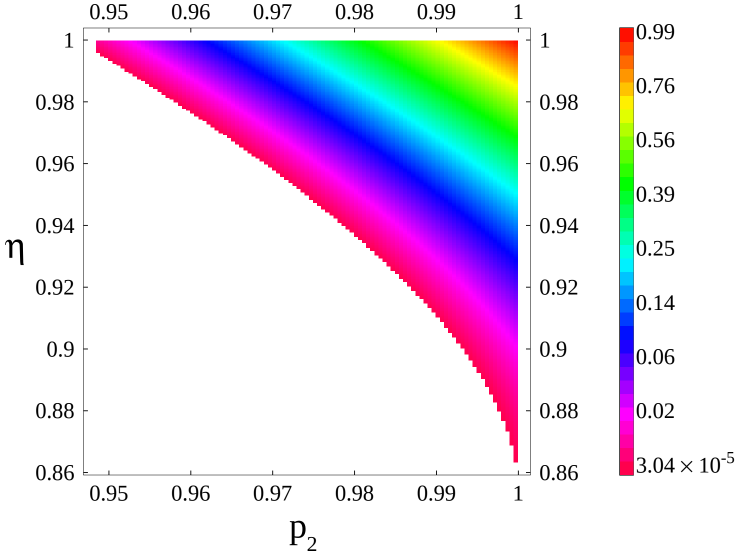
Poisson Random Graph at critical point $pN = 1$

Choose random pair of vertices. Entangle pair via purify-swap, or direct swap. What is average gain in final entanglement ?

- Giant cluster of mass $N^{2/3}$
- Density of shortest paths independent of L. So, as Werner param. $x \to 1$ long paths dominate.
- "Good" ranges of x overlap more for large L: $\Rightarrow$ integrate
- Each path-subpath occurs with probability $\approx 1/N^2$
- At fixed x, contributions are from $L \approx 1/(1-x)$. Four factors
  - $\approx L$ paths contribute near x
  - $\approx L$ sub-path lengths per path
  - $\approx L$ alternate paths per sub-path
  - $\approx L$ positions along path for sub,alt-path pair.
- Advantage of purify-swap over swap, averaged over network is

$$\Delta\overline{C} \sim \frac{K}{N^2(1-x)^4} \text{ for large } N \text{ small } 1-x, \quad (K\approx 6.5\times 10^{-5})$$

Lapeyre, Perseguers, Lewenstein, Acín, QIC 2012

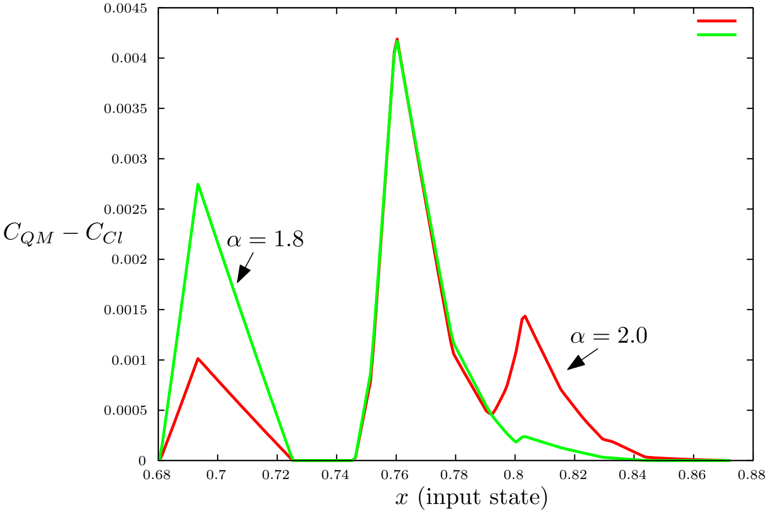


Noisy operations: $y = y_{\mathsf{max}}$, $a = a_{\mathsf{max}}(y_{\mathsf{max}})$, gives $\Delta C = 1/36$. η: reliability of measurement. p_2: reliability of two-qubit operator.

Poisson Random Graph at critical point $pN = 1$

But wait,...there's more. Return to perfect operations.

- For $Np = 1$, Radius grows like $N^{1/3}$ Nachmias, Peres, Ann. Prob. 2008
- Our MC shows radius of largest cluster $\approx 3N^{1/3}$.
- Since $L \approx 1/(1-x) \Rightarrow \Delta\overline{C} < 81AN^{-2/3}$
- Purification protocols always give modest results. They must be used iteratively.
- *But*, Choose bond density to favor $L = 2, 3$: $p^2N = c$. Then $\sigma_2 \rightarrow (1 - e^{-c})$ and $\sigma_3 \rightarrow e^{-c}$. Now for Werner parameter around 0.7, we have many subgraphs for purify-swap.



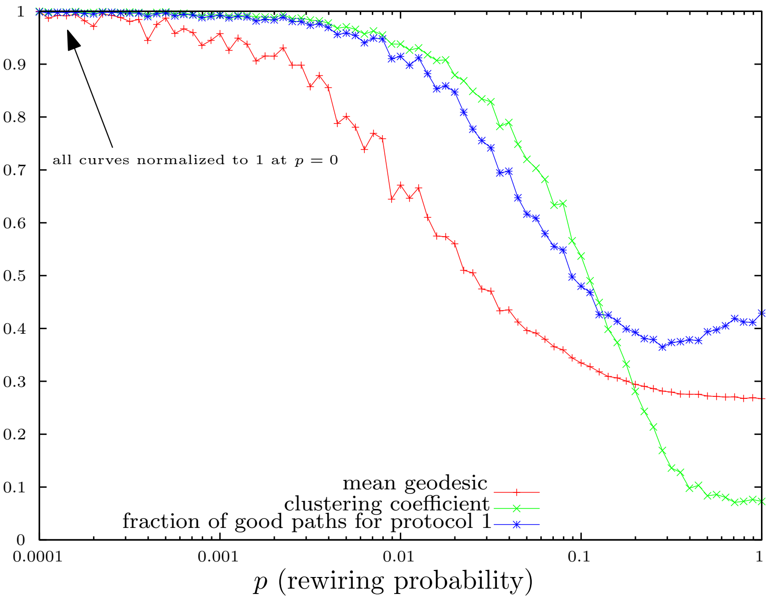
Advantage of purify-swap for Werner states on scale-free network with $p(k) \propto k^{-\alpha}$. $N = 200$.

Distribution of shortest paths on Watts-Strogatz model for $p = 0$. One link to each neighbor at $\pm 1, \pm 2$.

- $N(N-1)/2$ shortest paths (SPs)
- Number of SPs of each length L from 1 through $N/4 - 1$ is $2N$ (and $3N/2$ for boundary case $L = N/4$.)
- Density of SPs of length L is then $\sigma_L = 4/(N-1)$. Flat.(except for boundary case.)

Number of shortest paths admitting SPP (single purify-swap)

$$\frac{N^2 - N}{2} - 4N - \frac{1}{2} 2N(5) = \frac{N(N-19)}{2}.$$



Watts-Strogatz small world (2×2 neighbors). Fraction of paths admitting purify-swap. ($N = 100$).