

# Entanglement Distribution on Complex Networks

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Wilhelm und Else Heraeus-Seminar Classical and Quantum Transport  
in Complex Networks

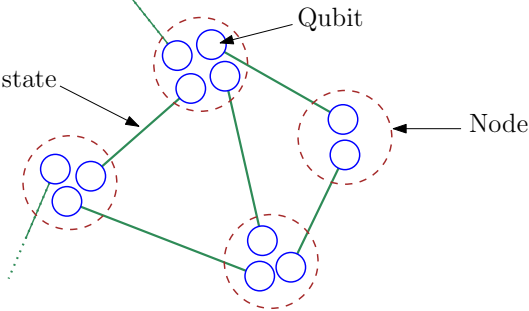


dynamics in complex systems

institute of physics



Bi-partite entangled state



- *qubits* (two-level quantum systems) (spin 1/2, photon polarization)
- Multiple qubits and classical resources at each *node* (vertex)
- *links* (edges): bi-partite entangled (pure/mixed) two-qubit states.
- Goal: entangle pairs of qubits between distant nodes
- Quantum operations local: within nodes. Classical can be global.  
Local Operations and Classical Communication **LOCC**

# Generalization of one-dimensional networks

- Quantum Information: Entanglement is a resource for tasks: teleportation, key distribution, fault tolerant computation
  - Creating entanglement requires local interaction. Noise increases with distance. Depolarization. Absorption. **Can't distribute entanglement over long distance in a single stage!**
- Long range entanglement via Network of stations or nodes that store and purify a state.
  - Generalization of **quantum repeater** schemes. Dür, Briegel, Cirac, Zoller, PRA 1999
  - Nodes share partially entangled states of qubits
  - Nodes(stations)/channels, Vertices/edges, Sites/bonds
  - Quantum operations **probabilistic**
  - Large number of random components  $\Rightarrow$  **Complex Networks, Percolation, Phase transition**

# Entanglement: Two entangled qubits



Two entangled qubits: four-dimensional Hilbert space.

## Bi-partite pure state

All such states LOCC equivalent to unique state in Schmidt basis.

$$|\alpha\rangle = \sqrt{\alpha_0} |00\rangle + \sqrt{\alpha_1} |11\rangle$$

$$\alpha_0 > \alpha_1 \quad \alpha_0 + \alpha_1 = 1 \quad \alpha_1 \in [0, 1/2]$$

Pure, partially entangled, bipartite state

$\alpha_1 = 0$ : no entanglement,  $\alpha_1 = 1/2$ : max. entanglement

# Bell State: Singlet Conversion



Partially Entangled:  $|\alpha\rangle = \sqrt{\alpha_0} |00\rangle + \sqrt{\alpha_1} |11\rangle$

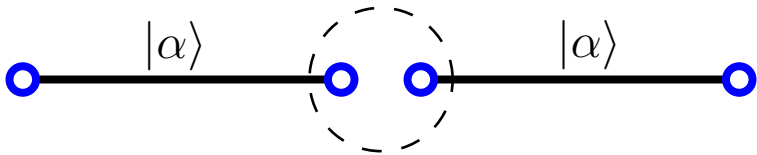
Local operations (and classical communication): qubits not allowed to interact

Maximally Entangled:  $|\Psi\rangle = \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle$

Singlet, Bell State, Maximally Entangled State **Singlet Conversion Probability**  $p = 2\alpha_1$ , for  $\alpha_0 > \alpha_1$

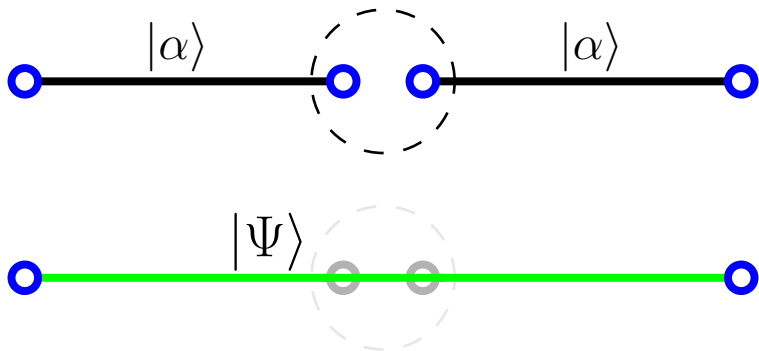
Otherwise: product state (failure)

# Entanglement Swapping



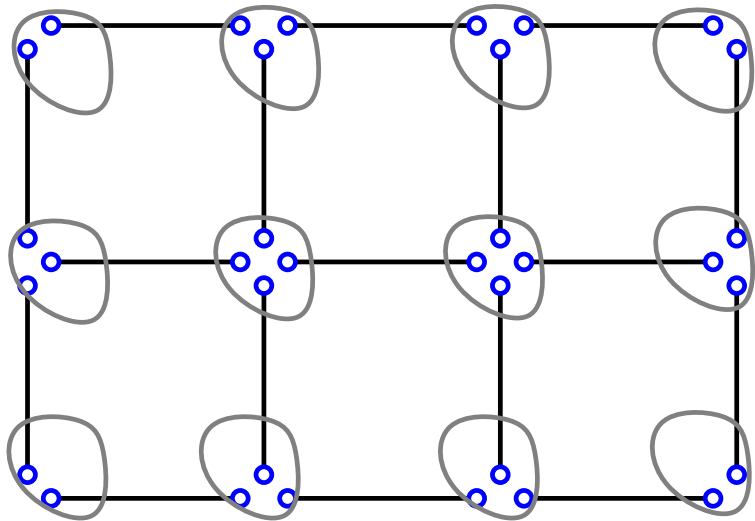
We can entangle the two outermost qubits, using only local operations and classical communication: *i.e.* without interacting outermost qubits. Using **entanglement swapping**.

# Entanglement Swapping



Entanglement Swapping. Get **Bell state** with same probability as in singlet conversion  $p = 2\alpha_1$  ! (product state otherwise) Note: if  $\alpha_1 = 1/2$ , then  $p = 1$ .

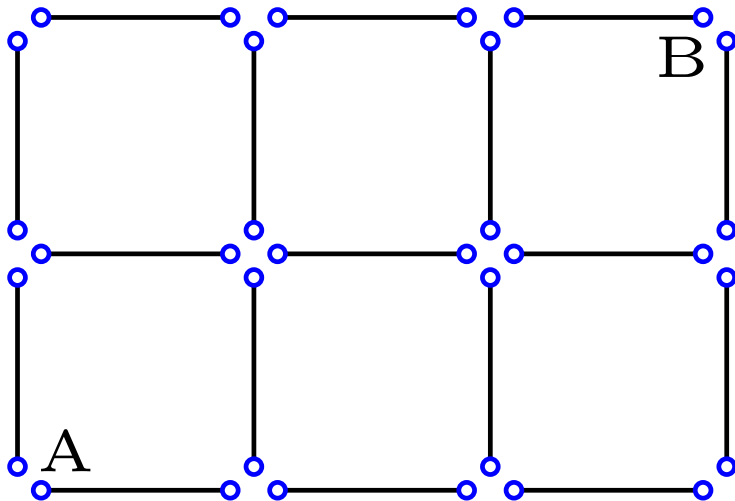
# Quantum Network



Concrete: Square lattice. Each bond is an entangled pair with amount of entanglement  $\alpha_1$ .

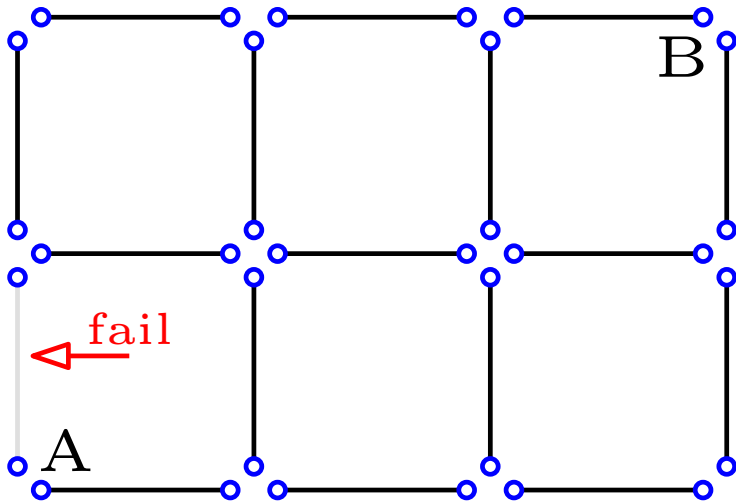


# "Classical" Entanglement Percolation

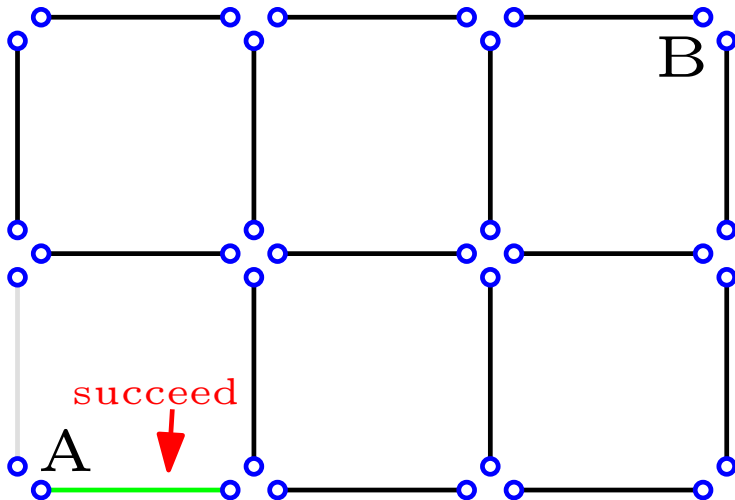


Entangle nodes A and B

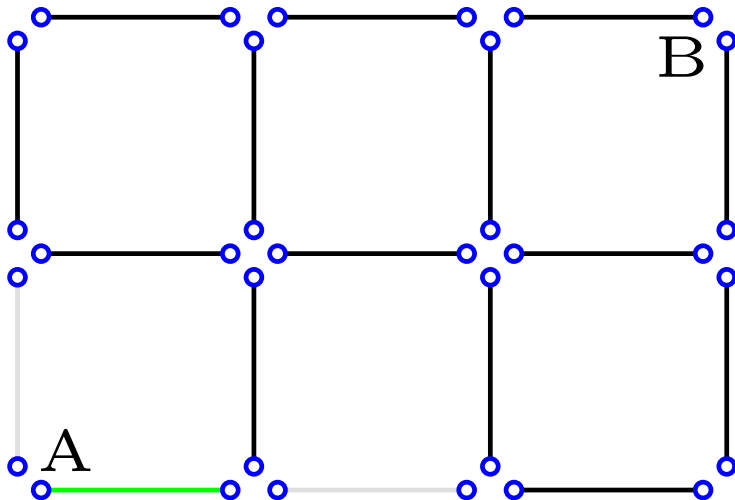
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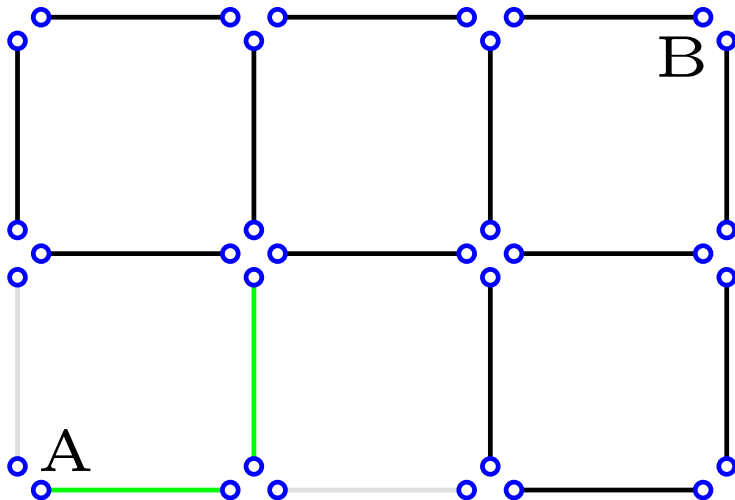
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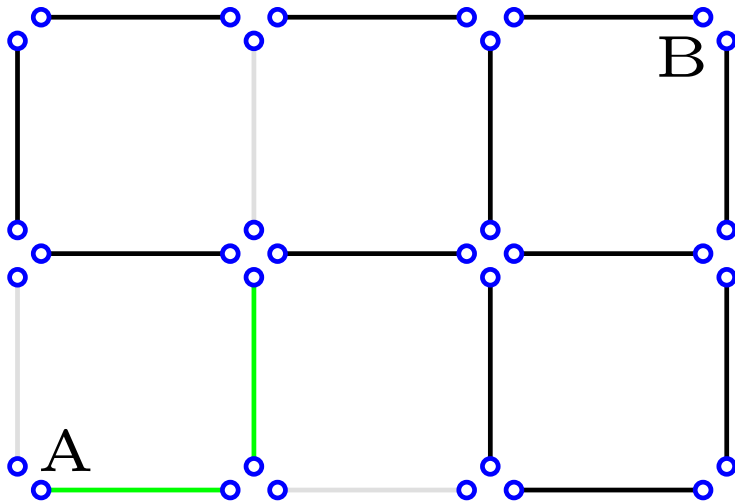
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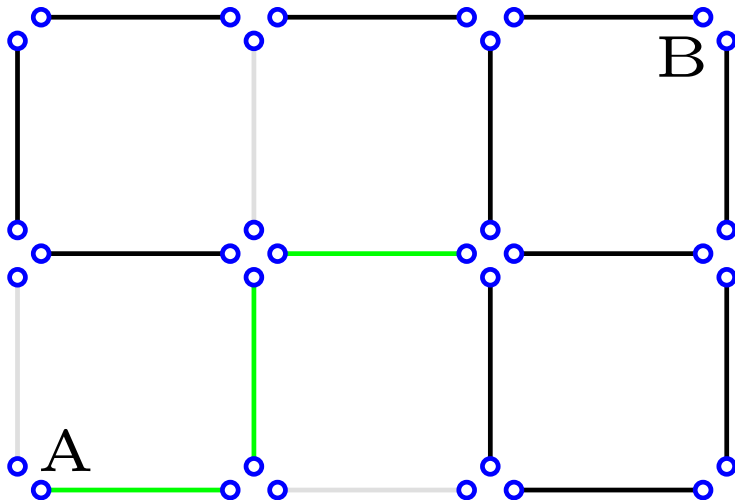
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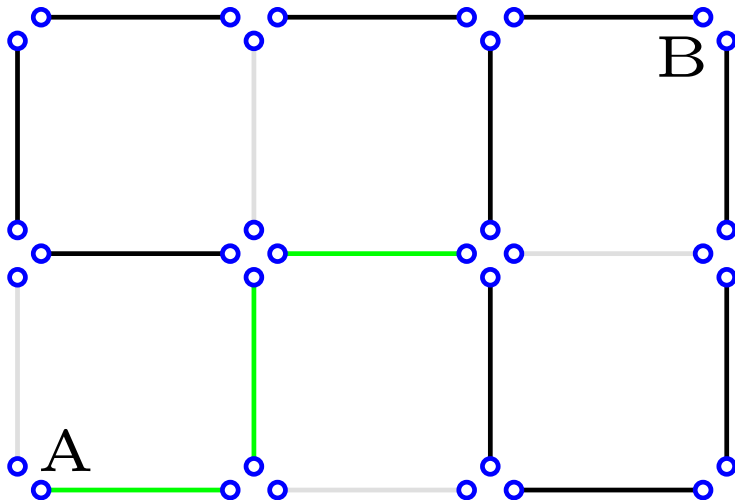
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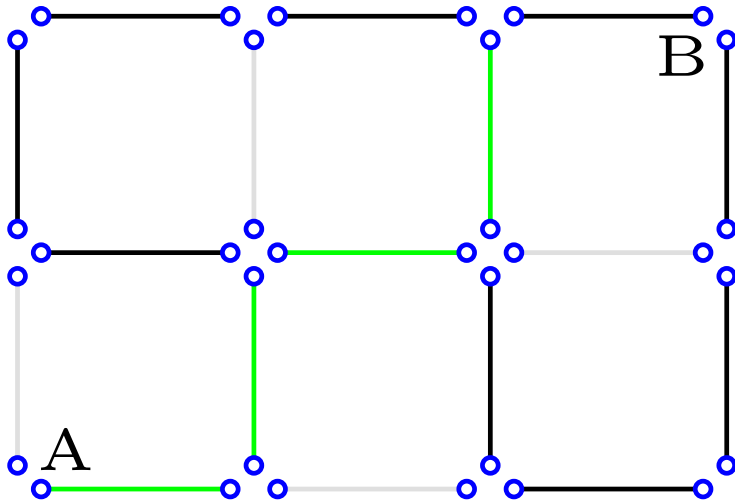


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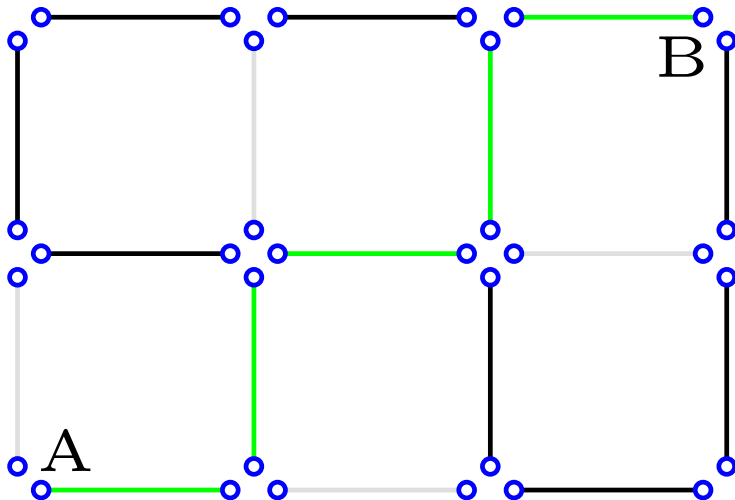




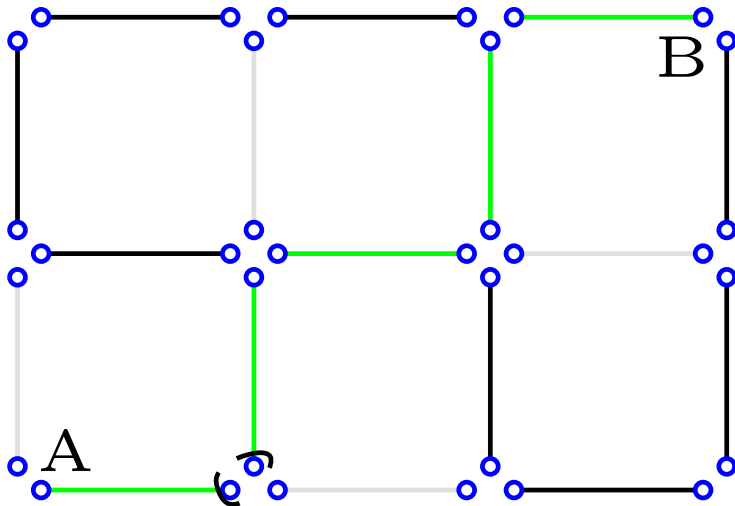
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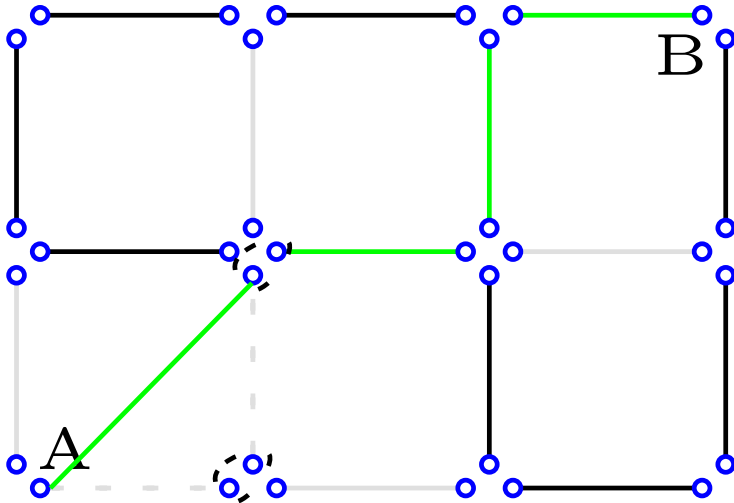
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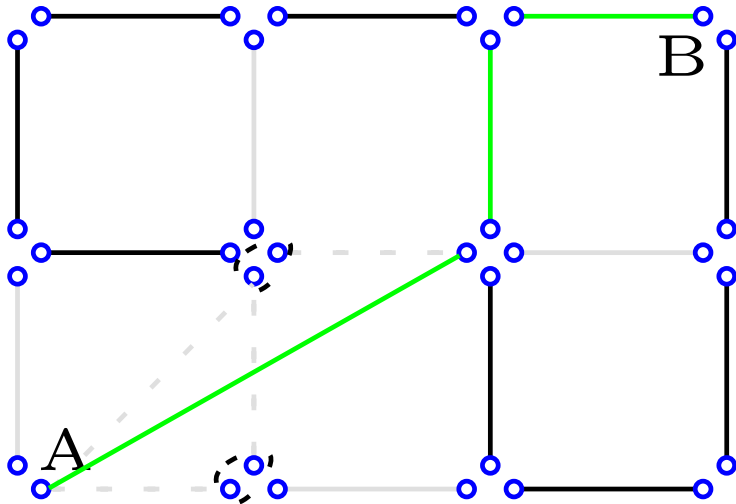
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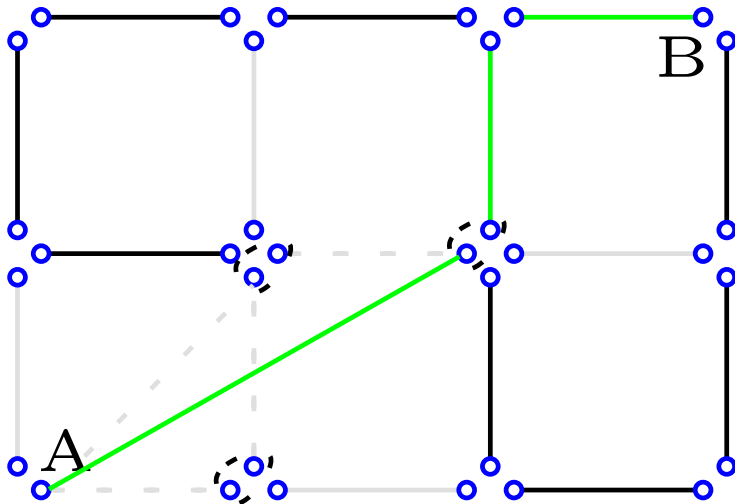
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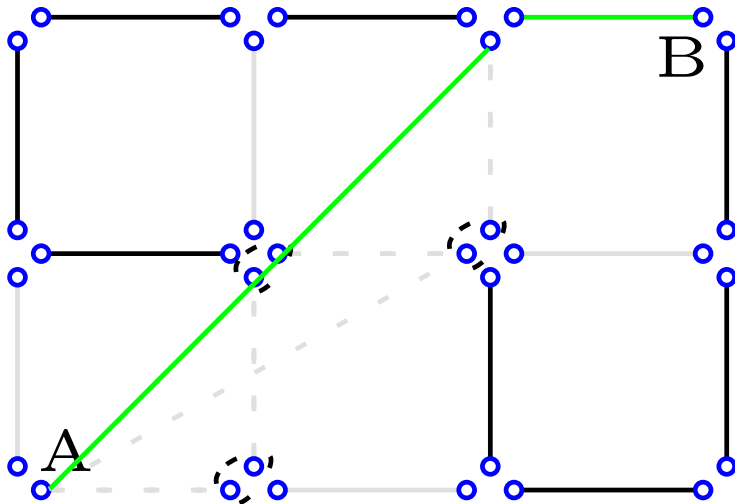
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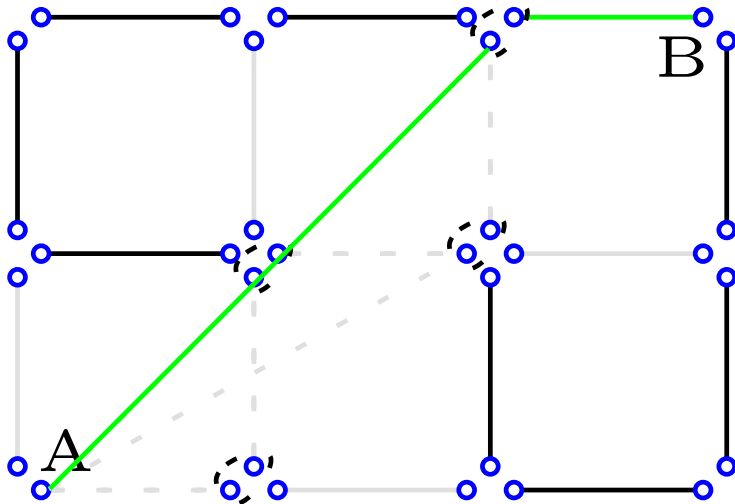
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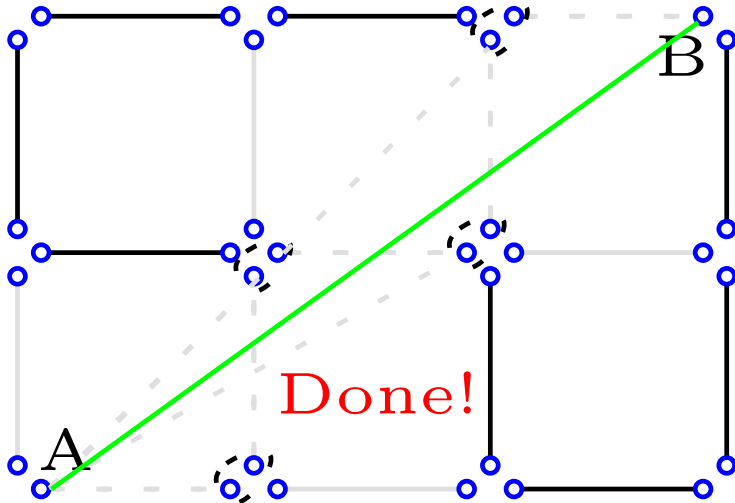


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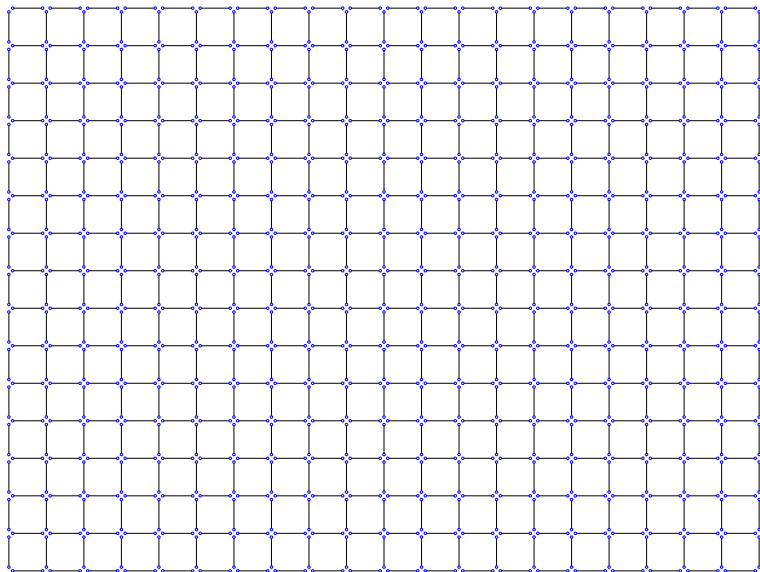




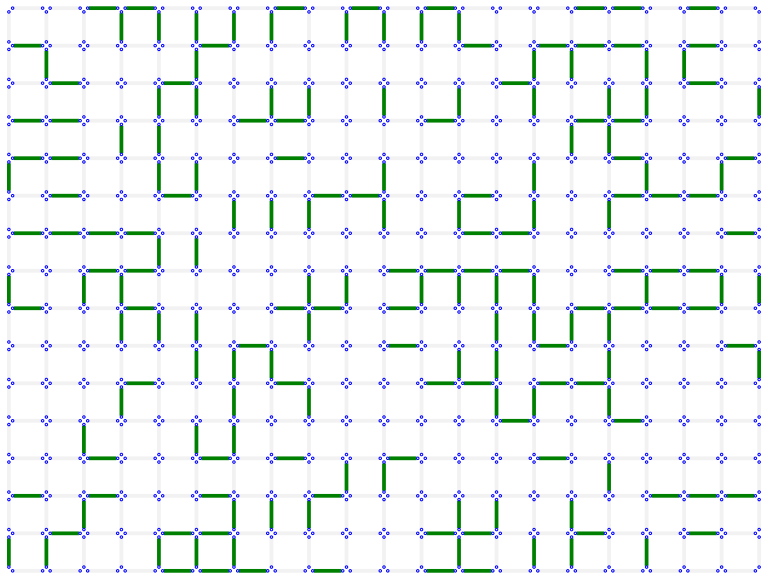
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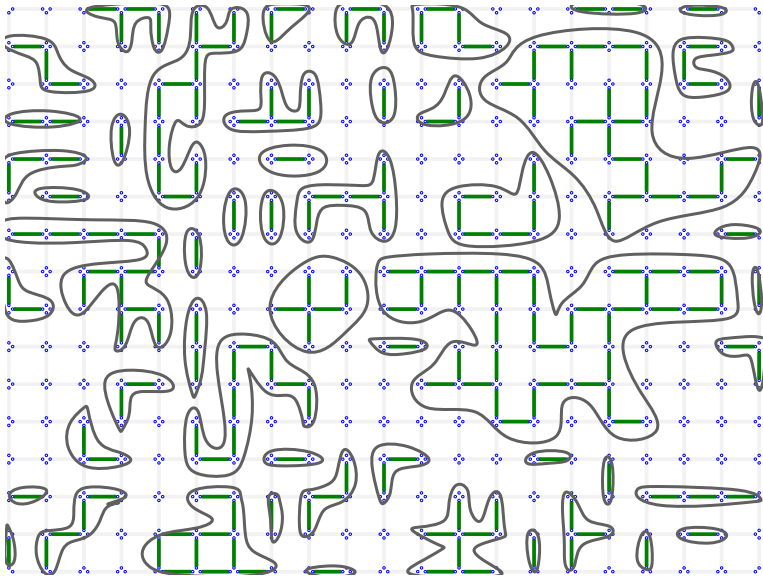
Big Network:  $\alpha_1 = 0.175$   $p = 0.35$



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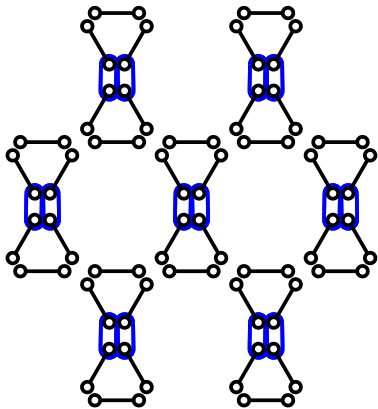


Big Network:  $\alpha_1 = 0.175$   $p = 0.35$

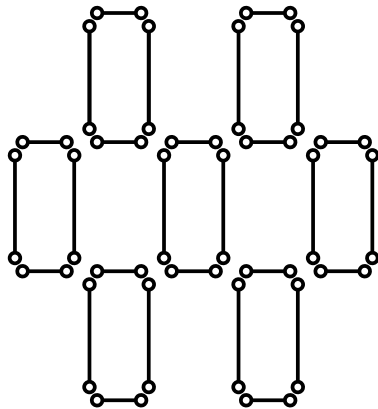


Percolation structure. Infinite cluster iff  $p > p_c = 0.5$  on square lattice.

## Kagome lattice to Square lattice



$p_c \approx 0.52$  Kagome

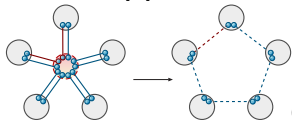


$p_c = 0.5$  Square lattice

## More entanglement percolation with pure states:

- Multipartite (GHZ) initial states  $\Rightarrow$  percolation on Archimedean and non-planar graphs. Perseguers, Cavalcanti, Lapeyre, Lewenstein, and Acín
- Improved swapping. Project onto larger subspace
- Conditionally complete swapping.
- Mixed states of rank  $\leq 3$  Broadfoot, Dorner, Jaksch, PRA 2010, EPL 2009
- Q-star transformation applied to various complex networks. E.g. for scale-free network, q-star usually advantageous when applied where degree is near

mean degree.



Cuquet, Calsamiglia, PRL 2009, PRA

2011

Let's leave these and **move to full-rank mixed states and complex networks.**

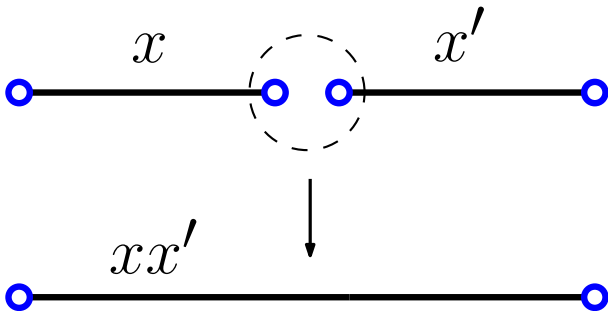
## Two-qubit Mixed States

- Full-rank mixed state. All four eigenvalues positive.
- Realistic noise. But, Cannot purify finite number of states to Bell pair (Jané QIC 2002)
- Two-qubit **Werner** (actually *isotropic*) state parameterized by  $x$

$$\rho_W(x) = x |\Phi_{00}\rangle\langle\Phi_{00}| + \frac{1-x}{4} \mathbb{1}_4, \quad 0 \leq x \leq 1$$

- Werner state is a full-rank state (for  $x < 1$ ).
- Separable for  $x \leq 1/3$ .
- **Concurrence**:  $C(x) = \max\{0, (3x - 1)/2\}$ . **Linear in  $x$** ,  
 $C(\text{separable, i.e. not entangled}) = 0$ ,  $C(\text{Bell pair}) = 1$

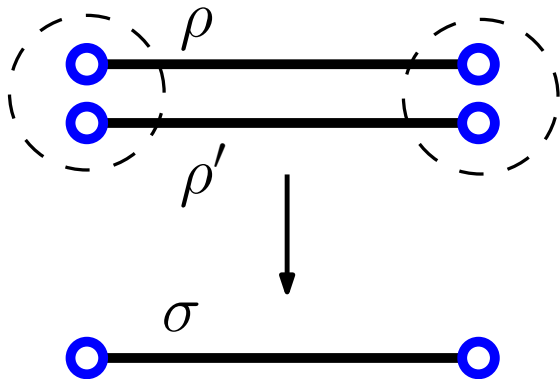
Swap Werner states. Get another Werner state.



Entanglement increases with  $x$ ; Exponential decay of entanglement with length of chain. Swapping: lose entanglement, Purification: gain entanglement.

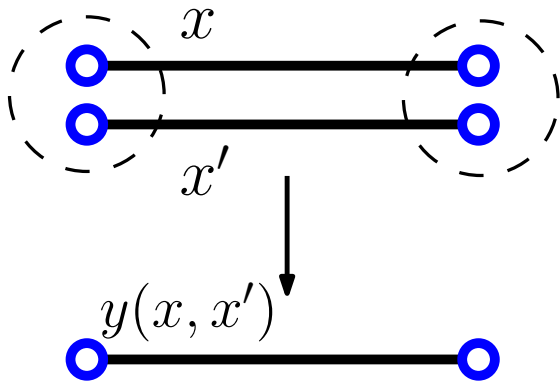


## Mixed states: Entanglement Purification



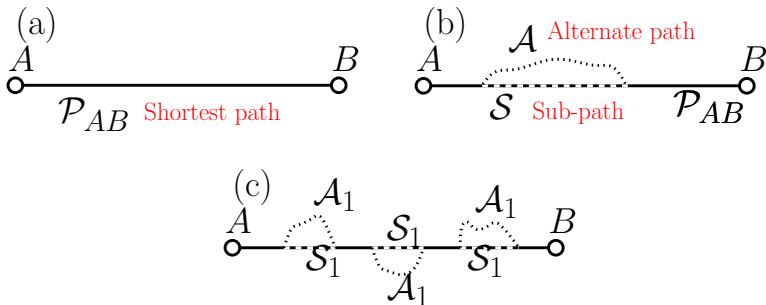
Obtain  $\sigma$  with entanglement greater than  $\rho, \rho'$  using LOCC  
(local operations and classical communication)

Purify Werner states. Get another Werner state.



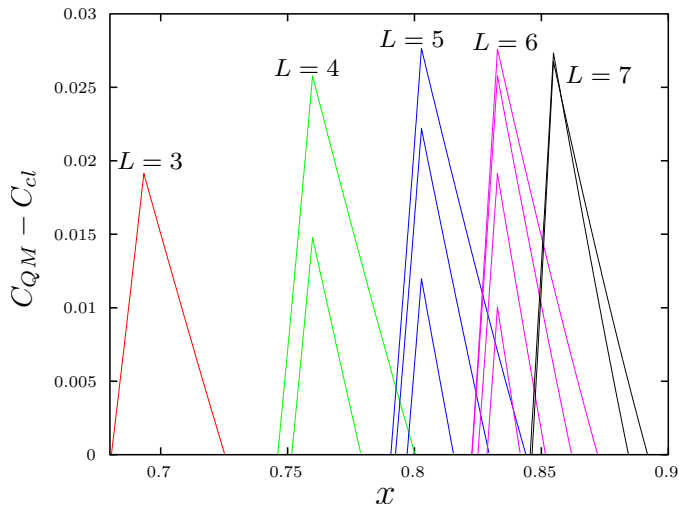
$$y(x, x') = \frac{x + x' + 4xx'}{3 + 3xx'}, \text{ with probability } \frac{1 + xx'}{2}$$

More generic network. Combine swapping and purification.



Swap first, or purify first? We call single-path purification **SPP**: 1) swap along sub-path; 2) swap along alternate path; 3) purify resulting states.

What is average concurrence? (over quantum outcomes)



Advantage of purify-swap depends on shortest path length  $L$ , sub-path length  $n$ , alternate path length  $m$ , Werner parameter  $x$ . Optimizing formula for gain in average concurrence is messy.

## Poisson Random Graph (or Erdős–Rényi Graph)

Apply **SPP** between random pairs of nodes.

- Graph with  $N$  vertices. Zero or one edge between each pair. Each of the  $N(N - 1)/2$  edges is present with probability  $p$ .
- Density of shortest paths of length  $L$ ,  $\sigma_L$

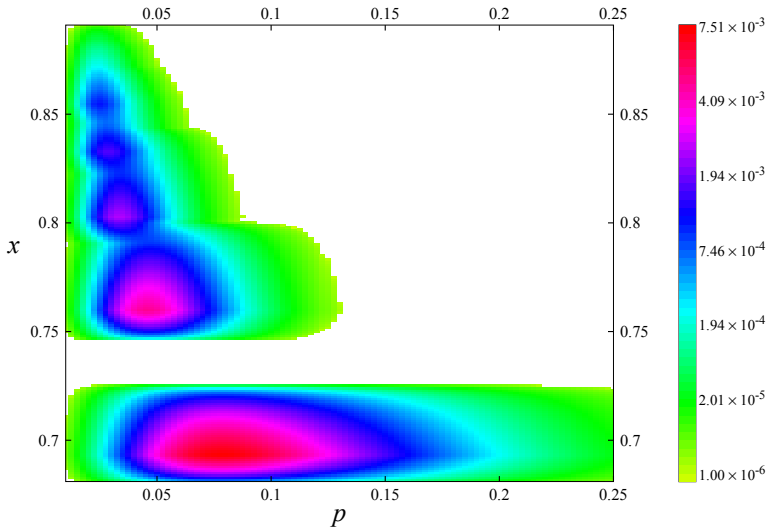
$$\sigma_1 = p,$$

$$\sigma_2 = (1 - p)[1 - (1 - p^2)^{N-2}] \approx (1 - p) \left(1 - e^{-p^2 N}\right),$$

$$\begin{aligned} \sigma_3 &\approx \left(1 - e^{-p^3(1-p)^5(N-2)(N-3)}\right) (1 - p^2)^{N-2}(1 - p), \quad \text{large } p \\ &\approx \left(1 - e^{-p^3(1-p)^5 N^2}\right) e^{-p^2 N} (1 - p) \end{aligned}$$

$$\sigma_L = p^L \frac{(N - 2)!}{(N - L - 1)!} + \mathcal{O}(p^{L+1}), \quad \text{small } p$$

$$\sigma_L \approx \frac{1}{N} \quad \text{for } pN = 1, \quad L < \text{radius}$$



Advantage of purify-swap depends on Werner parameter  $x$  and bond density of random graph  $p$ .

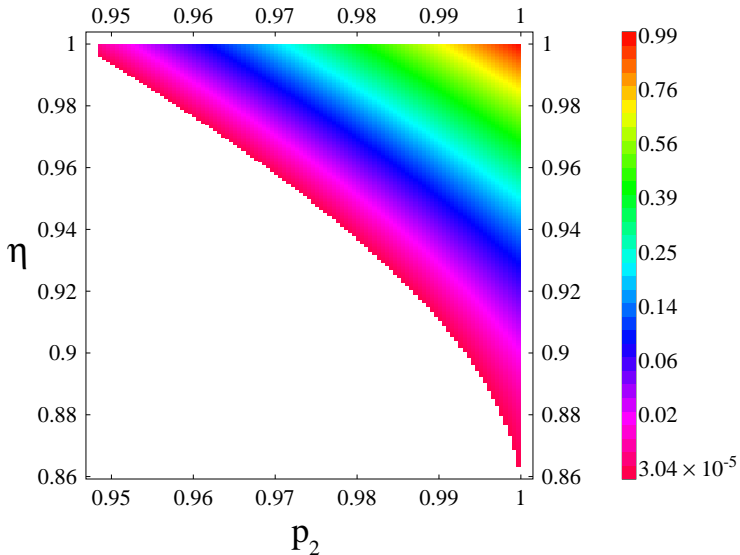
Monte Carlo  $N = 200$ ,  $L < 8$

## Poisson Random Graph at critical point $pN = 1$

Choose random pair of vertices. Entangle pair via purify-swap, or direct swap. What is average gain in final entanglement ?

- Giant cluster of mass  $N^{2/3}$
- Density of shortest paths independent of  $L$ . So, as Werner param.  $x \rightarrow 1$  long paths dominate.
- “Good” ranges of  $x$  overlap more for large  $L$ :  $\Rightarrow$  integrate
- Each path-subpath occurs with probability  $\approx 1/N^2$
- At fixed  $x$ , contributions are from  $L \approx 1/(1-x)$ . Four factors
  - $\approx L$  paths contribute near  $x$
  - $\approx L$  sub-path lengths per path
  - $\approx L$  alternate paths per sub-path
  - $\approx L$  positions along path for sub,alt-path pair.
- Advantage of purify-swap over swap, averaged over network is

$$\Delta \bar{C} \sim \frac{K}{N^2(1-x)^4} \text{ for large } N \text{ small } 1-x, \quad (K \approx 6.5 \times 10^{-5})$$



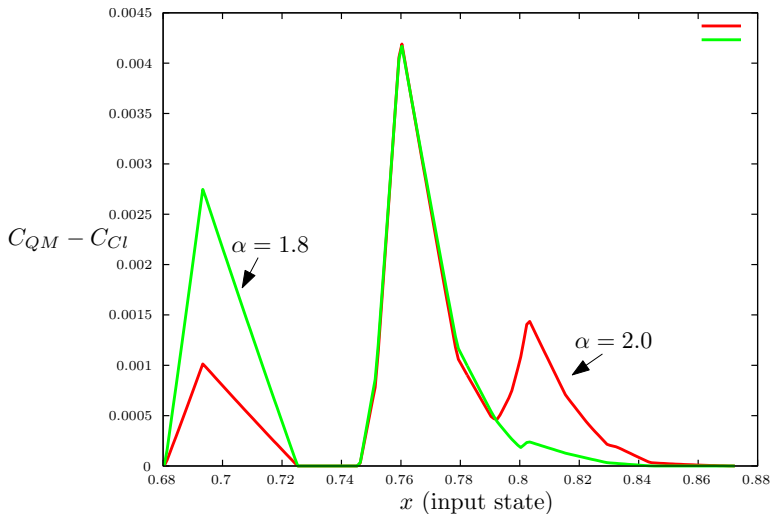
**Noisy operations:**  $y = y_{\max}$ ,  $a = a_{\max}(y_{\max})$ , gives  $\Delta C = 1/36$ .  $\eta$ : reliability of measurement.  $p_2$ : reliability of two-qubit operator.



## Poisson Random Graph at critical point $pN = 1$

But wait, . . . there's more. Return to perfect operations.

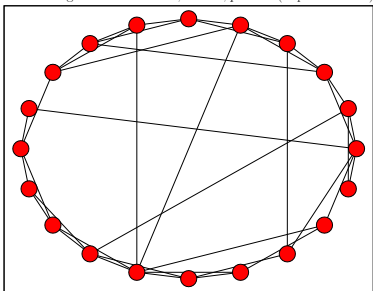
- For  $Np = 1$ , Radius grows like  $N^{1/3}$  Nachmias, Peres, Ann. Prob. 2008
- Our MC shows radius of largest cluster  $\approx 3N^{1/3}$ .
- Since  $L \approx 1/(1-x) \Rightarrow \Delta \bar{C} < 81AN^{-2/3}$
- Purification protocols always give modest results. They must be used iteratively.
- *But*, Choose bond density to favor  $L = 2, 3$ :  $p^2N = c$ . Then  $\sigma_2 \rightarrow (1 - e^{-c})$  and  $\sigma_3 \rightarrow e^{-c}$ . Now for Werner parameter around 0.7, we have many subgraphs for purify-swap.



Advantage of purify-swap for Werner states on scale-free network with  $p(k) \propto k^{-\alpha}$ . Monte Carlo with  $N = 200$ .

Watts-Strogatz model has parameter  $p$  that tunes between lattice-like model with regular local connections, and an Erdős–Rényi-like model. We compute distribution of shortest paths on Watts-Strogatz model for  $p = 0$ . One link to each neighbor at  $\pm 1, \pm 2$ , ( $K = 4$ ).

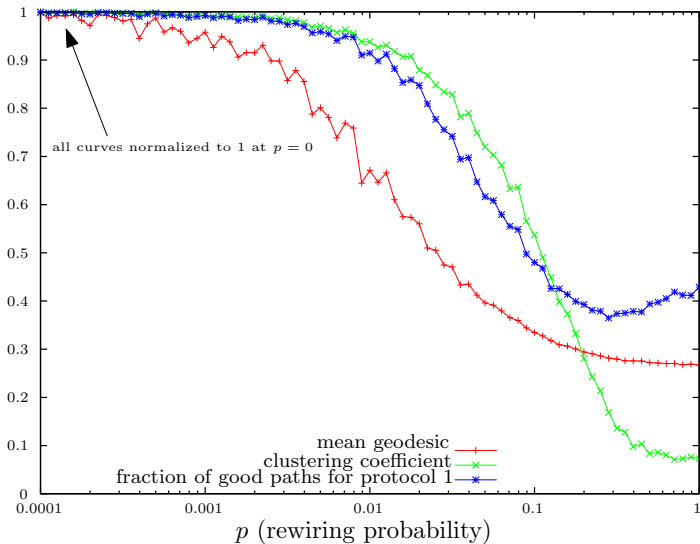
Watts-Strogatz model  $N = 20$ ,  $K = 4$ ,  $p = 0.2$  (Arpad Horvath)



- $N(N - 1)/2$  shortest paths (SPs)
- Number of SPs of each length  $L$  from 1 through  $N/4 - 1$  is  $2N$  (and  $3N/2$  for boundary case  $L = N/4$ .)
- Density of SPs of length  $L$  is then  $\sigma_L = 4/(N - 1)$ . Flat. (except for boundary case.)

Number of shortest paths admitting SPP (single purify-swap)

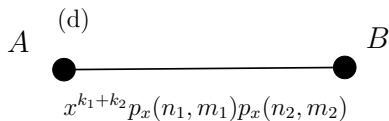
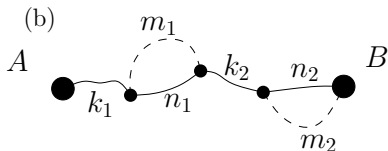
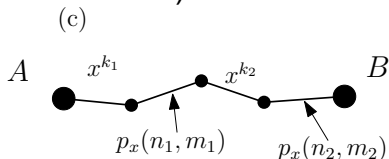
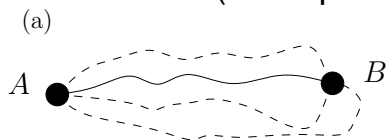
$$\frac{N^2 - N}{2} - 4N - \frac{1}{2}2N(5) = \frac{N(N - 19)}{2}.$$



Watts-Strogatz small world ( $2 \times 2$  neighbors). Fraction of paths admitting purify-swap. ( $N = 100$ ). The separation of the red and green lines is the main point of the model.

## Mixed states on complex network

Werner state on each link. What is average concurrence?  
(over quantum outcomes)



$p_x(n, m)$  is average Werner parameter after **SPP** with subpath length  $n$ , alternate path length  $m$ .