Entanglement Distribution on Complex Networks

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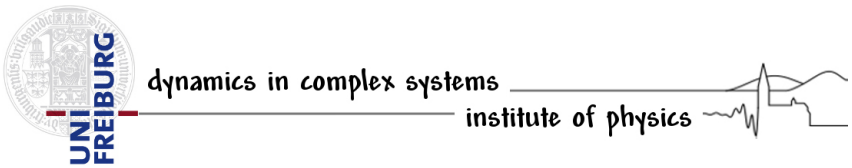
Wilhelm und Else Heraeus-Seminar Classical and Quantum Transport in Complex Networks



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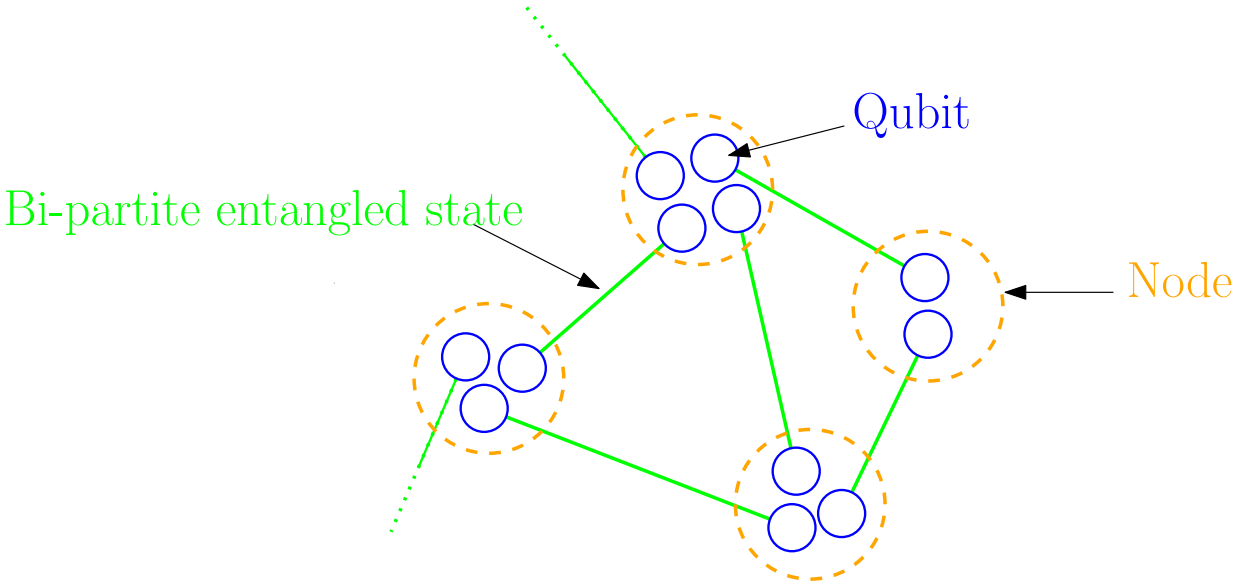


A **quantum network**.

- *qubits* (two-level quantum systems).
- Multiple qubits and classical resources at each *node* (vertex).
- *links* (edges): bi-partite entangled bi-partite entangled two-qubit states (pure/mixed).
- Quantum operations local: within nodes. Classical can be global. Local Operations and Classical Communication LOCC.
- Goal: entangle pairs of qubits between distant nodes.

Bell State: Singlet Conversion

Suppose links are partially entangled pure states. These can be converted (with probability) to maximally entangled pure states.

Partially Entangled: $|\alpha\rangle = \sqrt{\alpha_0}\,|00\rangle + \sqrt{\alpha_1}\,|11\rangle$

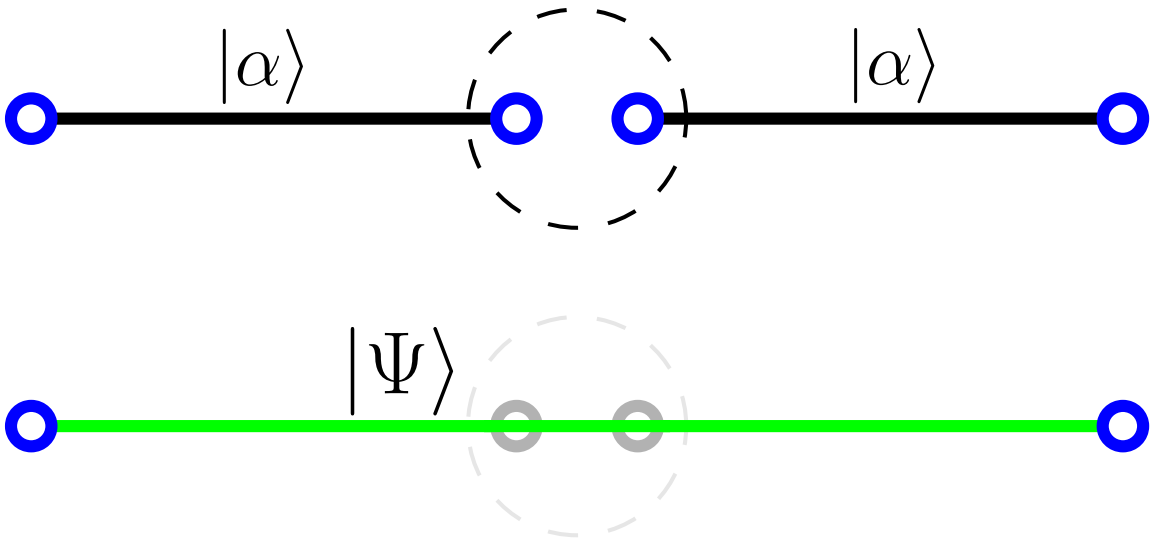
Local operations (and classical communication): qubits not allowed to interact

Maximally Entangled: $|\Psi\rangle = \frac{1}{\sqrt{2}}\,|00\rangle + \frac{1}{\sqrt{2}}\,|11\rangle$

Singlet, Bell State, Maximally Entangled State Singlet Conversion Probability $p = 2\alpha_1$, for $\alpha_0 > \alpha_1$

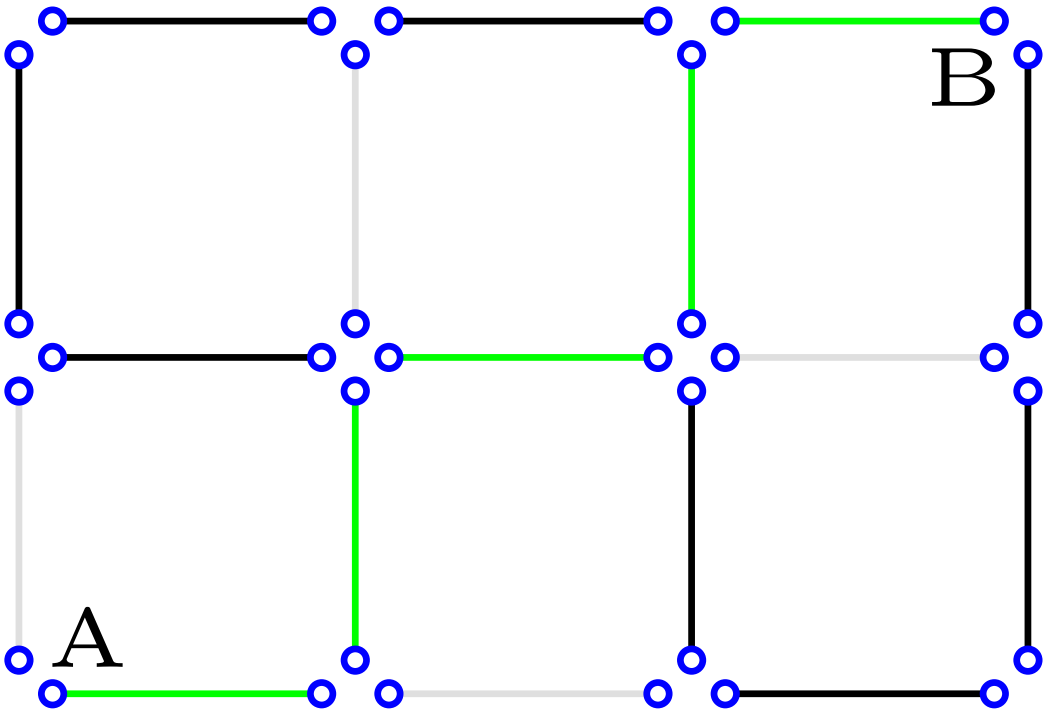
Otherwise: product state (failure)

Distributing Entanglement



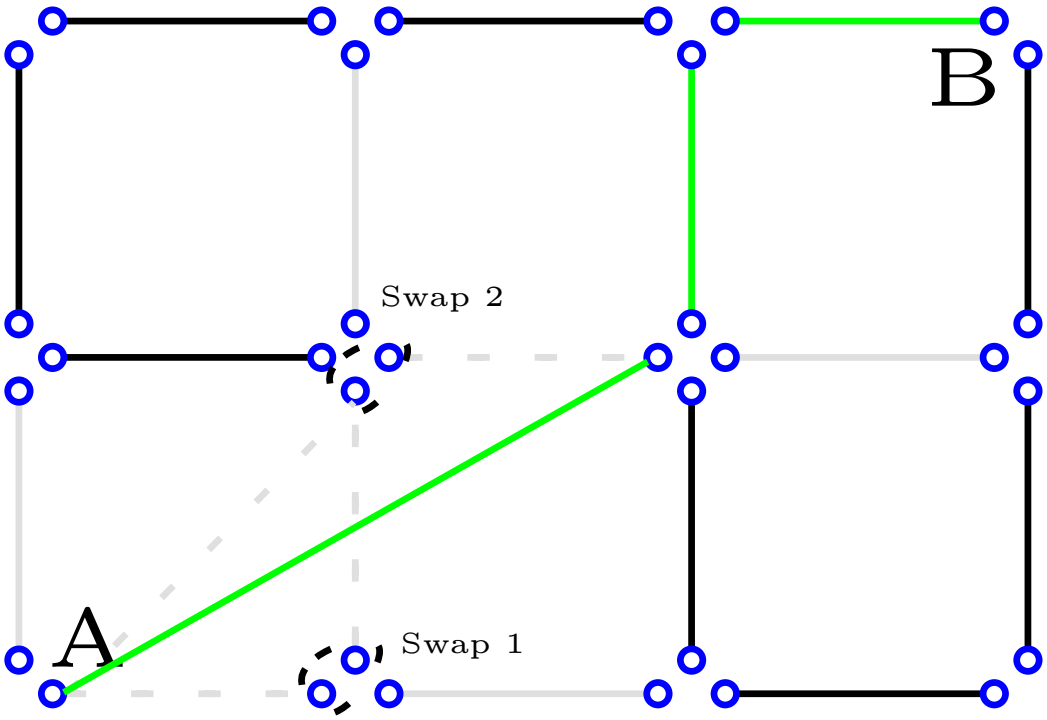
Second ingredient: Entanglement Swapping. We want to entangle the outermost qubits. Get Bell state with same probability as in singlet conversion $p = 2\alpha_1$! (product state otherwise) Note: if $\alpha_1 = 1/2$, then $p = 1$. Using only local operations and classical communication.

Classical Entanglement Percolation



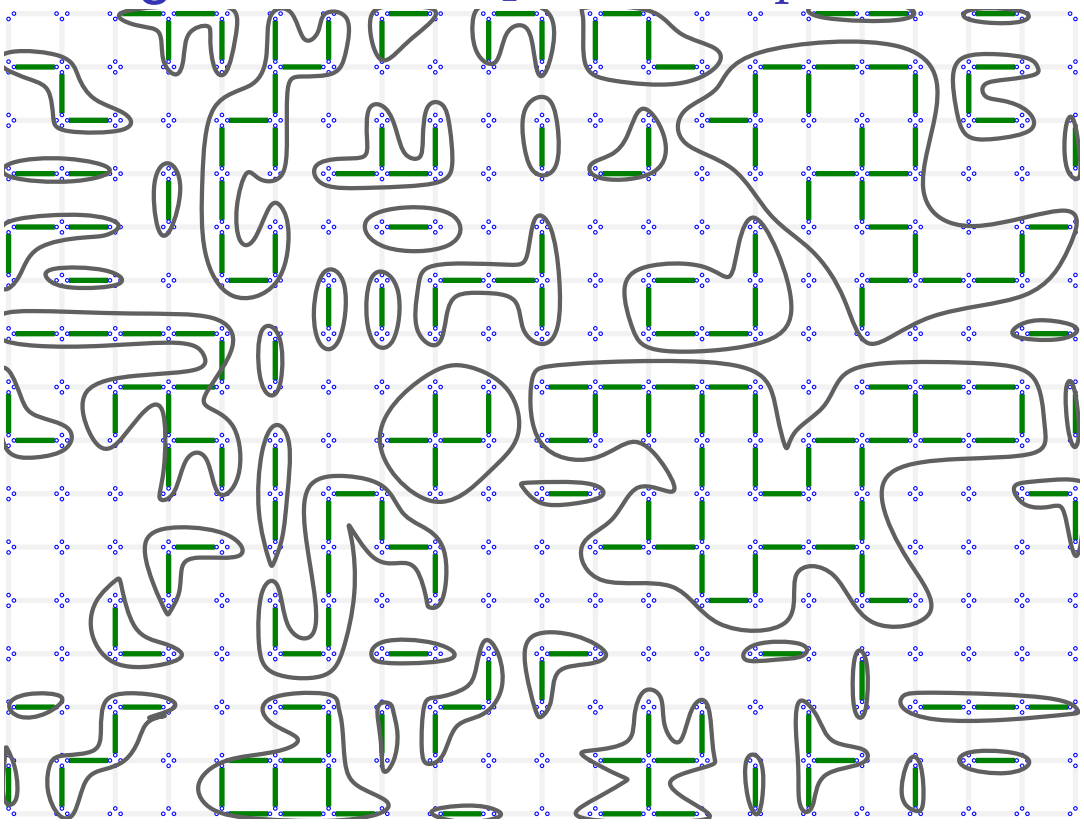
To entangle A and B, we start by doing singlet conversions everywhere. Here, green is success and gray is failure (separable).

Classical Entanglement Percolation



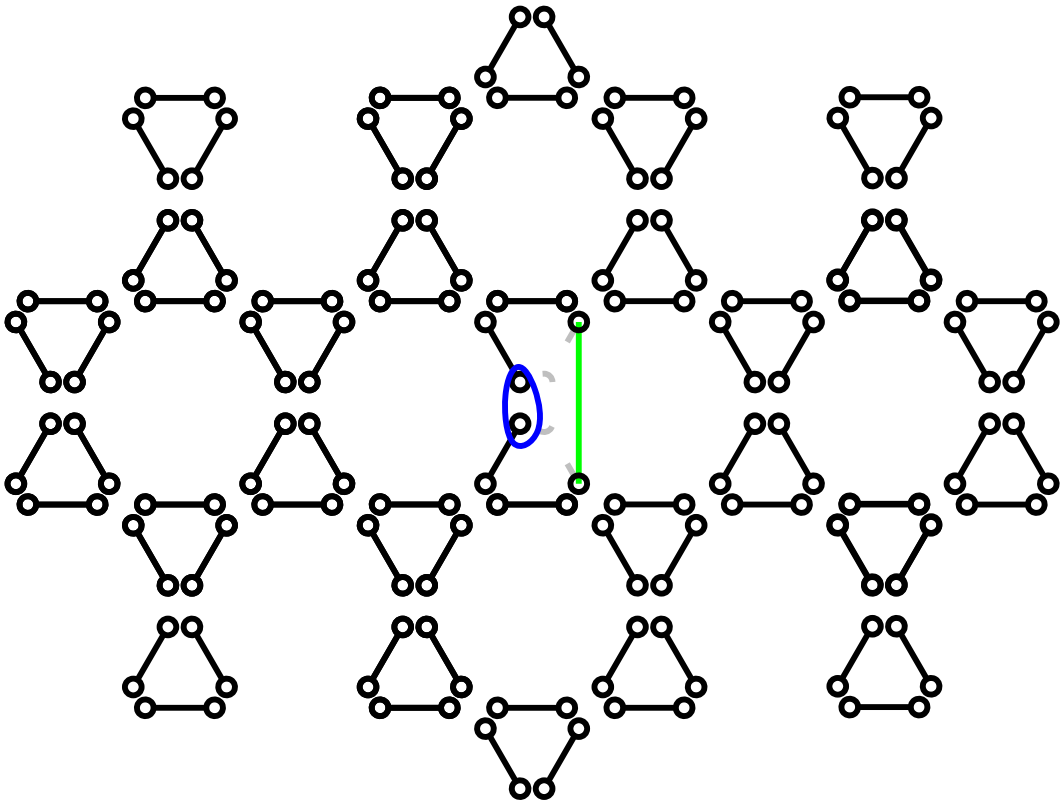
Then, entanglement swapping at every node with two Bell pairs.

Big Network: $\alpha_1 = 0.175 \;\; p = 0.35$



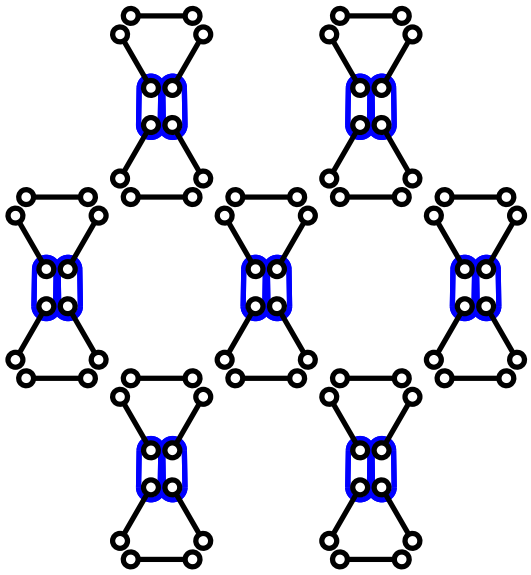
On a big network, singlet conversions map the model to percolation. Critical density $p_c = 1/2$ on square lattice. Long range entanglement only possible if $p > p_c$.

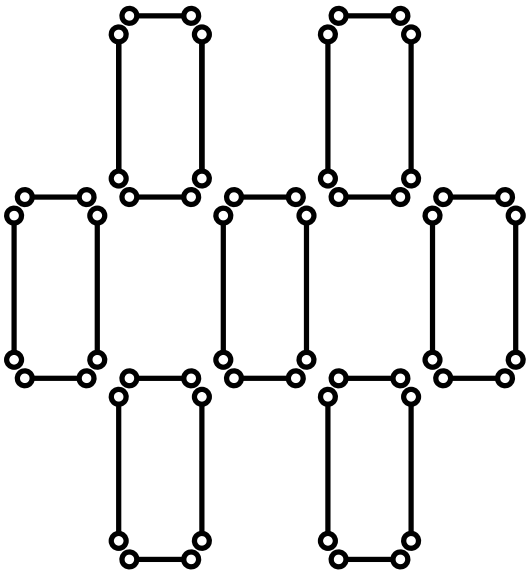
Kagome lattice



But, we can do better than naively doing singlet conversions first. The quickest example is the Kagome lattice. We first swap everywhere to create vertical bonds.

Kagome lattice to Square lattice





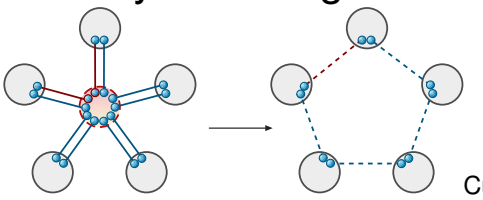
$p_c \approx 0.52$ Kagome $\qquad$ $p_c = 0.5$ Square lattice

Converts Kagome to square lattice. Then we proceed with swapping. For $0.5 < p < 0.52$, long range entanglement that was impossible in "classical" method becomes possible.

Acín, Cirac, Lewenstein, Nature Phys 2007
Perseguers, Cirac, Acín, Lewenstein, Wehr, PRA 2008 Lapeyre, Wehr, Lewenstein, PRA 2009

More entanglement percolation with pure states:

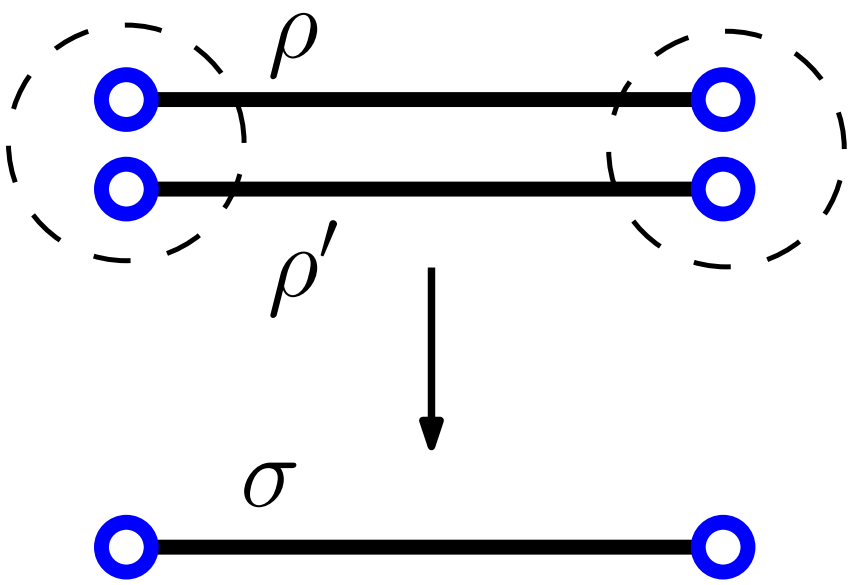
- Multipartite (GHZ) initial states $\Rightarrow$ percolation on Archimedean and non-planar graphs. Perseguers, Cavalcanti, Lapeyre, Lewenstein, and Acín
- Improved swapping. Project onto larger subspace. Preserves one qubit in center and allows multipartite entanglement.
- Only complete swapping conditioned on intermediate outcome. Allows more use of geometry.
- Mixed states of rank ≤ 3 Broadfoot, Dorner, Jaksch, PRA 2010, EPL 2009
- Q-star transformation applied to various complex networks. E.g. for scale-free network, q-star usually advantageous when applied where degree is near mean degree.



Cuquet, Calsamiglia, PRL 2009, PRA 2011

Let's leave these and move to full-rank mixed states and complex networks.

Mixed states: Entanglement Purification



Obtain σ with entanglement greater than ρ, ρ' using LOCC (local operations and classical communication)

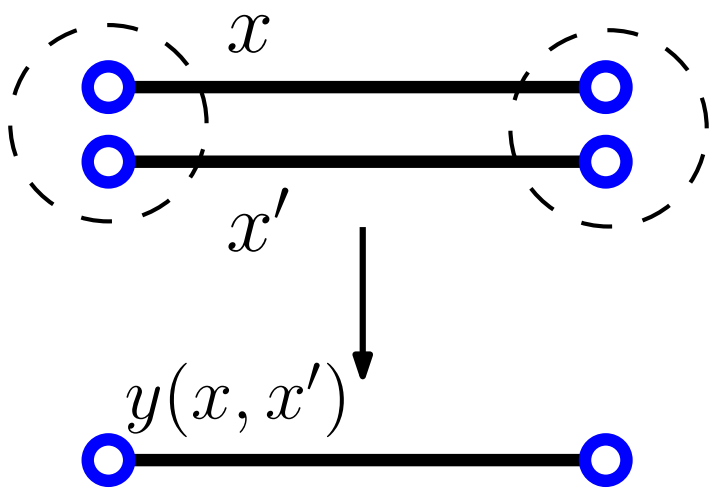
Two-qubit Mixed States

- Full-rank mixed state. All four eigenvalues positive.
- Cannot purify finite number of states to Bell pair
- Two-qubit Werner (actually "Isotropic") state parameterized by x

$$\rho_{\mathsf{W}}(x) = x\,|\Phi_{00}\rangle\langle\Phi_{00}| + \frac{1-x}{4}\mathbb{1}_4, \qquad 0 \le x \le 1$$

  - Werner state is a full-rank state (for $x < 1$).
  - Separable for $x \le 1/3$.
  - Concurrence: $C(x) = \max\{0, (3x-1)/2\}$. Linear, $C(\text{separable}) = 0$, $C(\text{Bell pair}) = 1$
  - Convert any state to $\rho_{\mathsf{W}}(x)$ via twirling. Can be done in lab. $\rho_{\mathsf{W}}(x)$ invariant under twirl.

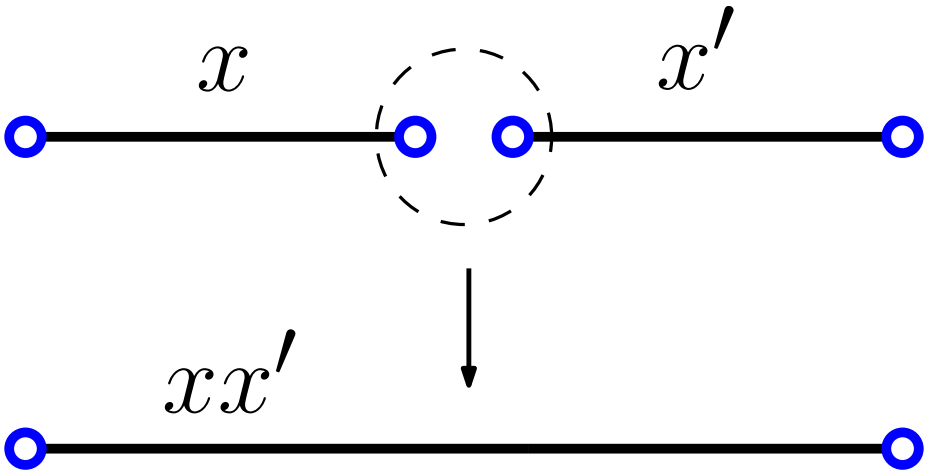
Purify Werner states. Get another Werner state.



$$y(x, x') = \frac{x + x' + 4xx'}{3 + 3xx'}, \text{ with probability } \frac{1 + xx'}{2}$$

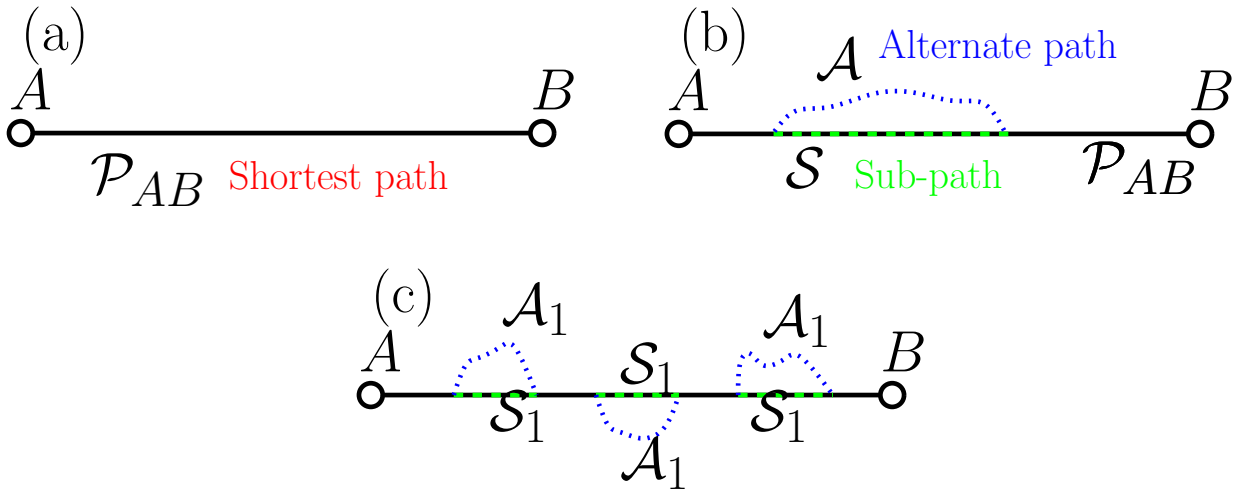
We choose analytically tractable purification protocol. But, input parameters x, x' must be similar to get increase. Bennett,Brassard,Popescu,Schumacher, PRL 1996 Dür, Briegel, Rep. Prog. Phys. 2007

Swap Werner states. Get another Werner state.



Output Werner parameter is xx'. Entanglement increases with x. So swapping results in an exponential decay of entanglement with length of chain. Swapping to distribute, we lose entanglement. Purifying to concentrate, we gain entanglement.

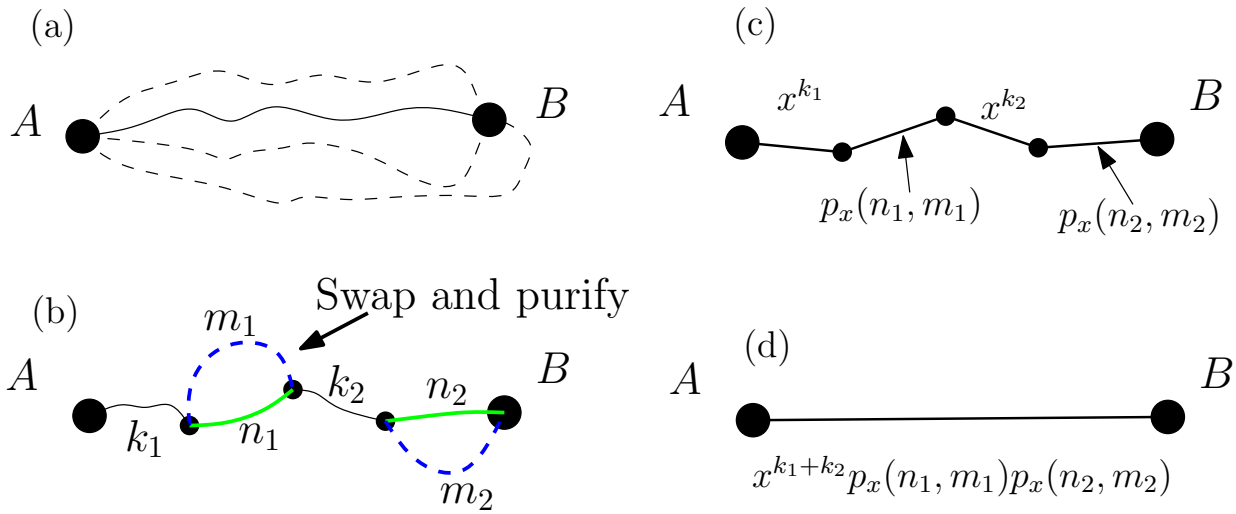
More generic network. Combine swapping and purification.



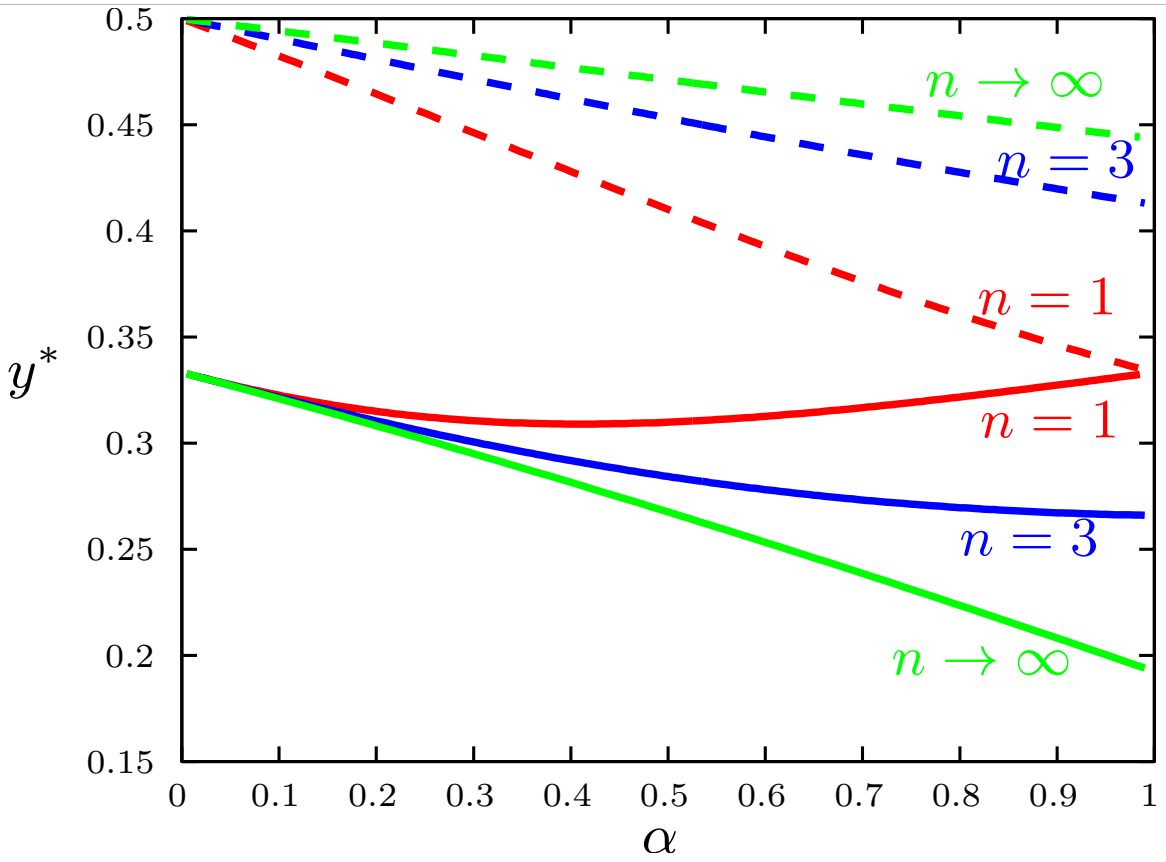
Swap first, or purify first? Do many purifications with short side chains, or fewer with longer chains? We call single-path purification SPP: 1) swap along sub-path; 2) swap along alternate path; 3) purify resulting states.

Mixed states on complex network

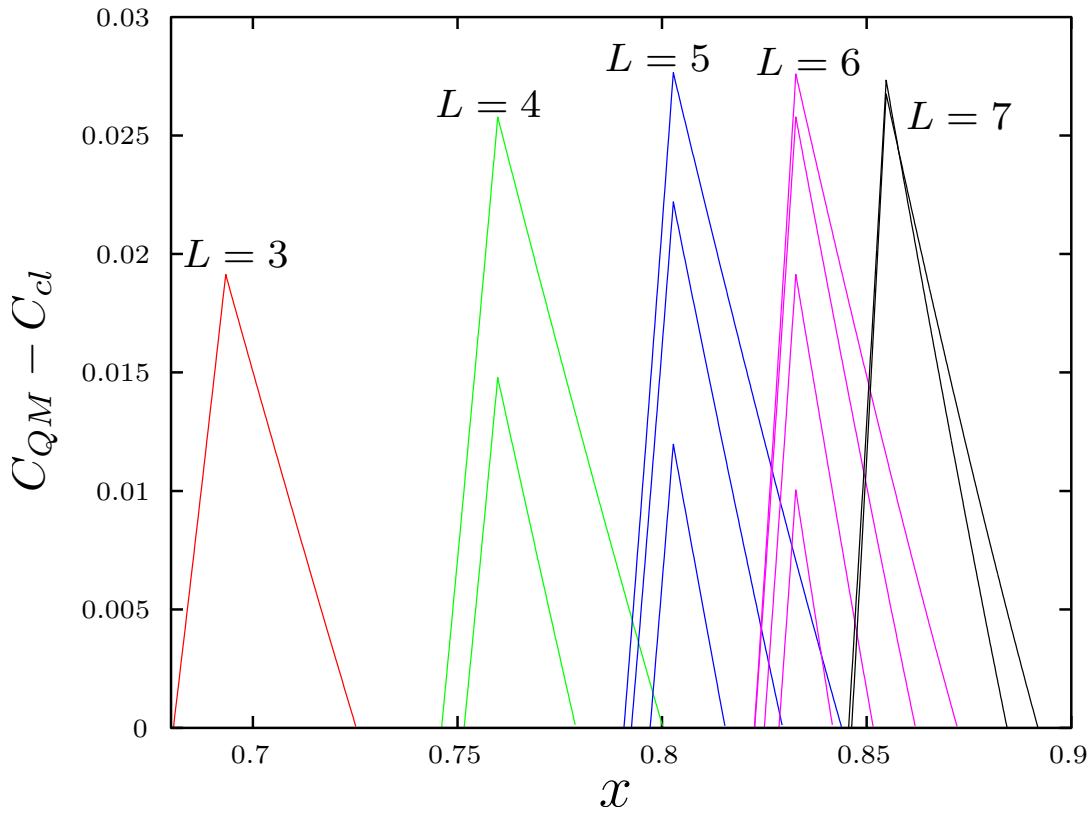
Werner state on each link. What is average concurrence? (over quantum outcomes)



$p_x(n, m)$ is average Werner parameter after SPP with sub-path length n, alternate path length m.



Multiple purifications along shortest path. Rescaled $y = x^{1/L}$; α is fraction of shortest path covered by subpaths in panel 15. Lines bound range of y for which purify-swap is good. Doing many swap-purifies is best, with utility tending to a limit. $y = (1/3)^{3/2} \approx 0.19$ for for $\alpha = 1$ and $n \to \infty$.



Advantage of purify-swap depends on shortest path length L, sub-path length n, alternate path length m, Werner parameter x. Optimizing formula for gain in average concurrence is messy.

Poisson Random Graph (or Erdös–Rényi Graph)

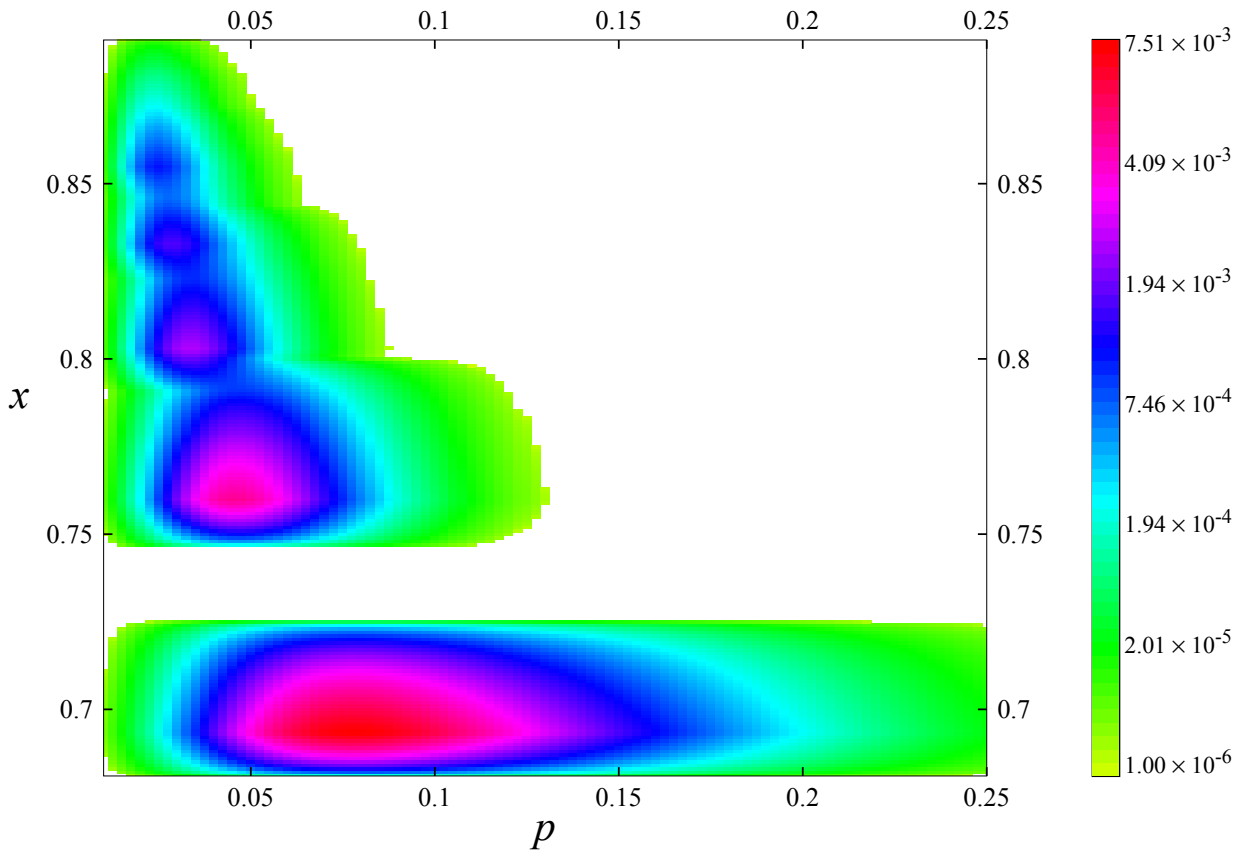
Apply single swap-purify to a random graph. Probability of occurrence of configurations as in panel 15 is computed from probability of shortest path lengths.

- Graph with N vertices. Zero or one edge between each pair. Each of the $N(N-1)/2$ edges is present with probability p.
- We call σ_L the density of shortest paths of length L.

$$
\begin{aligned}
\sigma_1 =& p, \\
\sigma_2 =& (1-p)[1-(1-p^2)^{N-2}] \approx (1-p)\left(1-e^{-p^2 N}\right), \\
\sigma_3 \approx& \left(1-e^{-p^3(1-p)^5(N-2)(N-3)}\right)(1-p^2)^{N-2}(1-p), \quad \text{large } p \\
\approx& \left(1-e^{-p^3(1-p)^5 N^2}\right) e^{-p^2 N}(1-p) \\
\sigma_L =& p^L \frac{(N-2)!}{(N-L-1)!} + \mathcal{O}(p^{L+1}), \quad \text{small } p \\
\sigma_L \approx& \frac{1}{N} \quad \text{for } pN = 1,\ L < \text{ radius}
\end{aligned}
$$

Density of SPP graphs can be computed in various limits using σ_L.

Solomonoff, Rapoport, Bull. Math. Biophys. 1948 (Who has these ?), 1951 Erdös, Rényi 1959,1960,1961



Advantage of purify-swap depends on Werner parameter x and bond density of random graph p. Red means best advantage over using no concentration.

Monte Carlo $N = 200$, $L < 8$

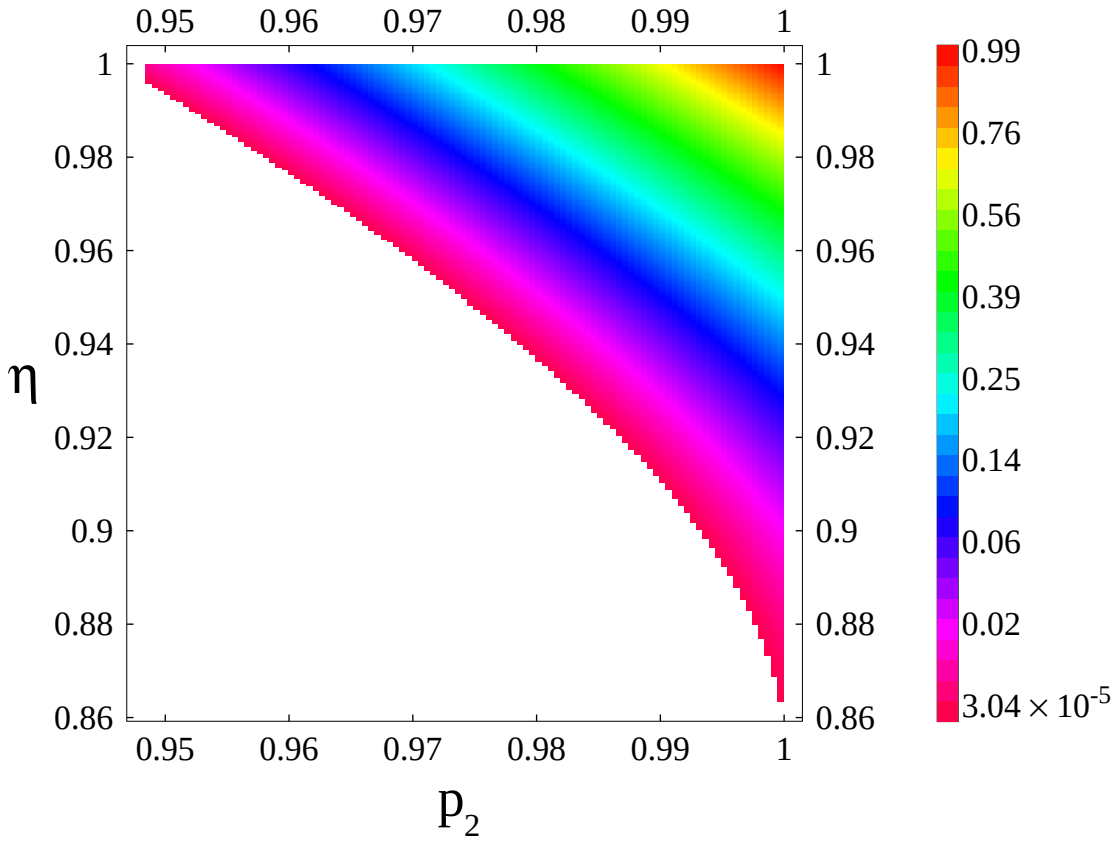
Poisson Random Graph at critical point $pN = 1$

Choose random pair of vertices. Entangle pair via purify-swap, or direct swap. What is average gain in final entanglement ?

- Giant cluster of mass $N^{2/3}$
- Density of shortest paths independent of L. So, as Werner param. $x \to 1$ long paths dominate.
- "Good" ranges of x overlap more for large L. So, we rewrite sum over SPP configurations as integral and get exact asymptotic result.
- Each path/sub-path occurs with probability $\approx 1/N^2$
- At fixed x, contributions are from $L \approx 1/(1-x)$. Four factors
  - $\approx L$ paths contribute near x
  - $\approx L$ sub-path lengths per path
  - $\approx L$ alternate paths per sub-path
  - $\approx L$ positions along path for sub,alt-path pair.
- Advantage of purify-swap over no concentration, averaged over network is

$$\Delta\overline{C} \sim \frac{K}{N^2(1-x)^4} \text{ for large } N \text{ small } 1-x, \quad (K \approx 6.5 \times 10^{-5})$$

Lapeyre, Perseguers, Lewenstein, Acín, QIC 2012



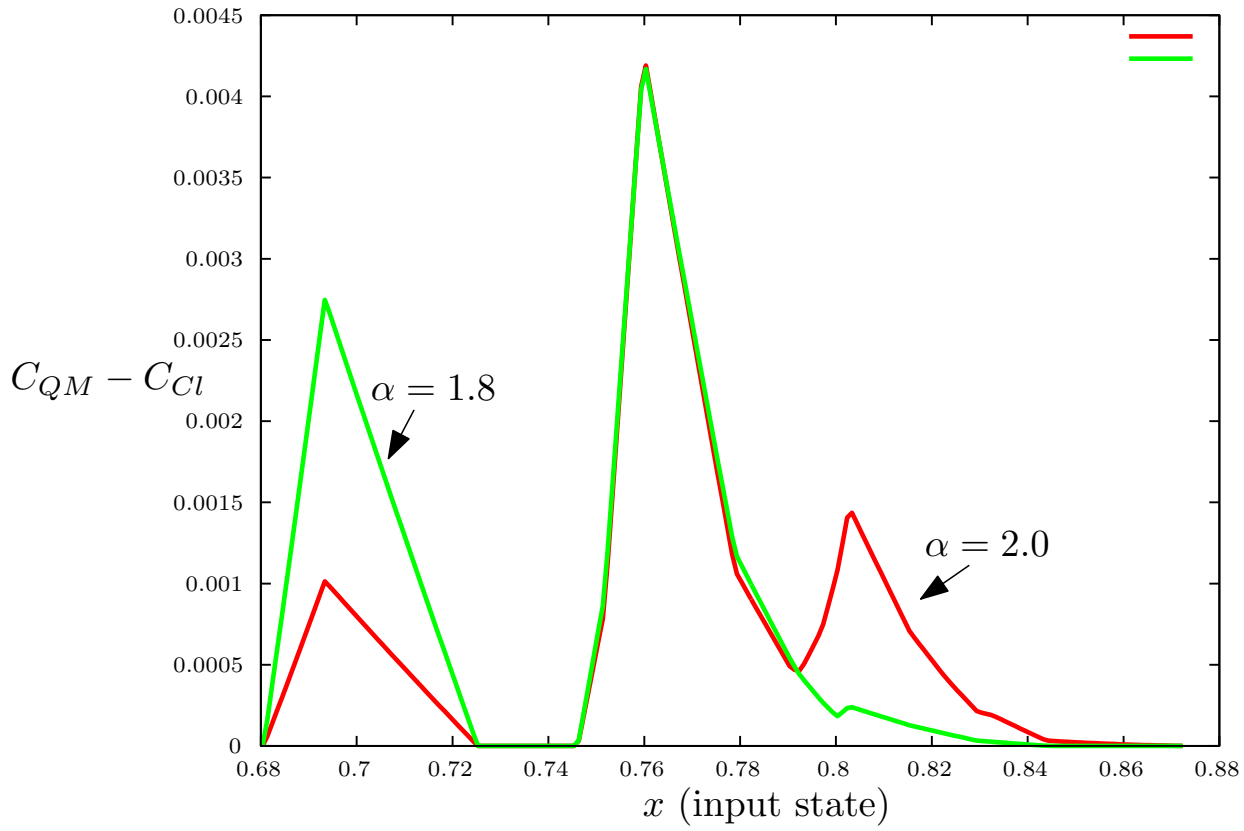
'Til now, noisy states, perfect operations. What if we use Noisy operations?

Choose optimal y, a for ΔC are $y = y_{\max}$, $a = a_{\max}(y_{\max})$: gives $\Delta C = 1/36$ with perfect operations. η is the reliability of measurement. p_2 is the reliability of two-qubit operator.

Poisson Random Graph at critical point $pN = 1$

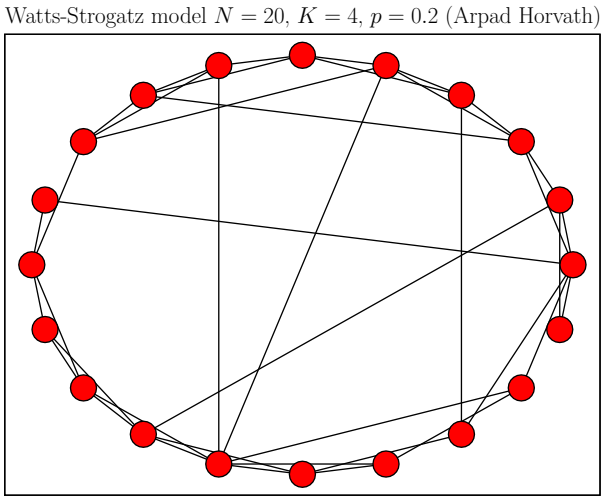
But wait,. . . there's more. Look again at perfect operations on noisy states. Probability of shortest path is independent of length only for L less than radius.

- For $Np = 1$, Radius grows like $N^{1/3}$ Nachmias, Peres, Ann. Prob. 2008
- Our MC shows radius of largest cluster $\approx 3N^{1/3}$.
- Since $L \approx 1/(1-x) \;\Rightarrow\; \Delta\overline{C} < 81AN^{-2/3}$
- Purification protocols always give modest results. They must be used iteratively. We need to swap with more side paths.
- *But*, we can choose bond density to favor $L = 2, 3$: $p^2N = c$. Then, from panel 19, $\sigma_2 \to (1 - e^{-c})$ and $\sigma_3 \to e^{-c}$. Now for Werner parameter around 0.7, we have many subgraphs for purify-swap.



Advantage of purify-swap for Werner states on scale-free network with $p(k) \propto k^{-\alpha}$. Monte Carlo with $N = 200$.

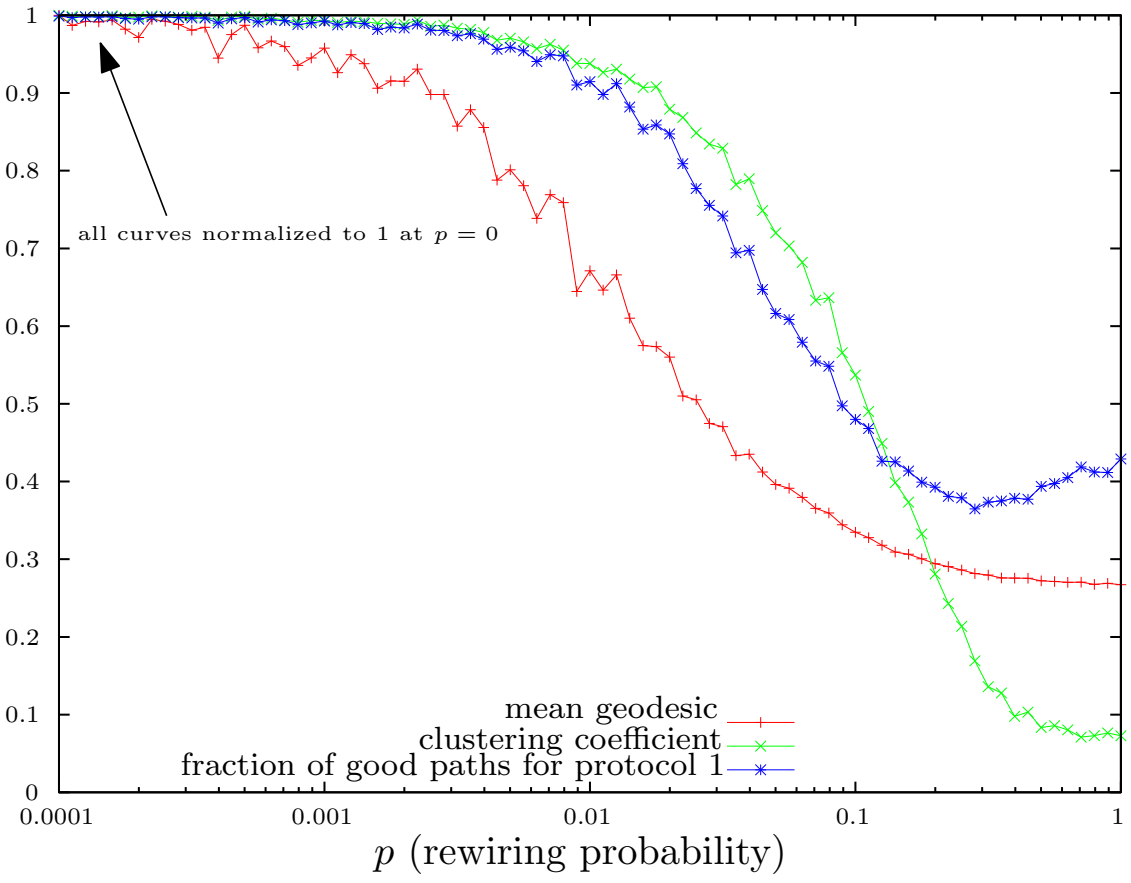
Watts-Strogatz model has parameter p that tunes between lattice-like model with regular local connections, and an Erdös–Rényi–like model. We compute distribution of shortest paths on Watts-Strogatz model for $p = 0$. One link to each neighbor at $\pm 1, \pm 2$, ($K = 4$).



- $N(N-1)/2$ shortest paths (SPs)
- Number of SPs of each length L from 1 through $N/4 - 1$ is $2N$ (and $3N/2$ for boundary case $L = N/4$.)
- Density of SPs of length L is then $\sigma_L = 4/(N-1)$. Flat.(except for boundary case.)

Number of shortest paths admitting SPP (single purify-swap)

$$\frac{N^2 - N}{2} - 4N - \frac{1}{2}2N(5) = \frac{N(N-19)}{2}.$$



Watts-Strogatz small world (2×2 neighbors). Fraction of paths admitting purify-swap. ($N = 100$). The separation of the red and green lines is the main point of the model. Paths admitting purify-swap follow one, then the other.